2 Short Questions

1. **(10 points)** The energies of an electron in a hydrogen atom are given by

$$E_n = -\frac{13.606 \text{ eV}}{n^2}$$

where $n = 1, 2, 3, \ldots$ represents the principal quantum number of the shell in which the electron is located.

The Ly-α spectral line is produced when an electron transitions from the $n = 2$ to the $n = 1$ energy level. Astronomers observe that the wavelength of the Ly-α line in a distant receding galaxy's emission spectrum is $\Delta\lambda = 7.13$ nm greater than the value measured in a lab.

Calculate the object's approximate distance from us in Mpc (assuming Hubble's constant $H_0 = 70$ km/s/Mpc).

Solution: To find the wavelength of the spectral line, we note that

$$\frac{hc}{\lambda} = E_2 - E_1$$

Then,

$$\lambda = \frac{hc}{E_2 - E_1} \approx 121.52 \text{ nm}$$

Answers within ± 0.5 nm of this result should be accepted due to potential differences in rounding.

In order to calculate the recessional velocity, we use the fact that $\Delta\lambda$ results from the redshift due to motion away from us. Thus,

$$v_r = cz = \frac{\Delta\lambda}{\lambda} c \approx 17600 \text{ km/s}$$

To calculate the distance to this object, we use Hubble's Law:

$$d = \frac{v_r}{H_0} \approx \boxed{251 \text{ Mpc}}$$

2. **(10 points)** The following expression describes the mass function of a binary system:

$$f(M_1, M_2) = \frac{M_2^3 sin^3(i)}{(M_1 + M_2)^2}$$

- M_1: Mass of star 1.
- M_2: Mass of star 2.
- i: Inclination of the orbit.

Consider an **eclipsing** binary system with a period of 70 years and a total semi-major axis of 36 AU. In this system, the semi-major axis of star 1 is two times larger than the semi-major axis of star 2.

Estimate the mass function of the binary system in terms of solar masses.

Solution:

The first step is to calculate the total mass of the system:

$$\frac{T^2}{a^3} = \frac{1}{M_{total}}$$

$$M_{total} = \frac{36^3}{70^2}$$

$$M_{total} = 9.5\ M_{\odot}$$

Since the semi-major axis of star 1 is two times larger than the semi-major axis of star 2, the mass of star 2 must be two times larger than the mass of star 1:

$$M_1 + M_2 = M_{total}$$

$$\frac{1}{2}M_2 + M_2 = 9.5\ M_{\odot}$$

$$M_2 = 6.3\ M_{\odot}$$

Since the binary system is eclipsing, the value of the inclination must be extremely close to 90°. Therefore, using this value in the calculations will result in a very accurate value for the mass function:

$$f(M_1, M_2) = \frac{M_2^3 sin^3(i)}{(M_1 + M_2)^2}$$

$$f(M_1, M_2) = \frac{6.3^3 sin^3(90^\circ)}{9.5^2}$$

$$\boxed{f(M_1, M_2) = 2.8\ M_{\odot}}$$

3. **(10 points)** Consider a star A (apparent magnitude $m_A = 10.9$, radius $R_A = 0.42R_{\odot}$). A periodic transiting event is observed to have a decrease the collected flux by 0.07 %. If this event was caused by a transiting exoplanet around star A, what would be the radius of that exoplanet in Earth radii?

Solution: The depth of a transit δ equals the ratio of the squared planet to star radii $\frac{R_p^2}{R_*^2}$. So, $R_p = \sqrt{\delta} \times R_* = \sqrt{0.0007} \times 0.42R_{\odot} = 1.212R_{\oplus}$.

4. **(10 points)** Posidonious from the first century BC estimated the circumference of the Earth by observing the rising and setting of the star Canopus. We will retrace his calculations in this problem. He observed Canopus on but never above the horizon at Rhodes. On the other hand, Canopus rose to a maximum of about 7.5° above the horizon at Alexandria. Assume Rhodes and Alexandria have the same longitude and the distance between the two cities is 800 km. Given only this information, estimate the radius of the Earth. How far off is it from the actual value of 6400 km. Justify your answer.

Solution: Given the simplifying assumption that the two cities belong to the same longitude, the difference of 7.5 ° in the maximum culmination altitude of Canopus, directly translates to a latitude difference between the two cities.

We then have, if d is the distance between Rhodes and Alexandria,

$$d = \frac{7.5}{360} 2\pi R_E$$

Hence, we get $R_E = 6111\,\text{km}$. This is a difference of about 5 %.

5. **(10 points)** There is an electron with its mass m_e that orbits a proton with mass m_p at a radius r. If we only assume the Coloumbic attraction,

(a) Write an expression of the total energy and the orbital momentum of the electron.

(b) Rewrite the expression of the total energy E in terms of the orbital momentum L, both from the part(a).

Use e for the electric charge quantity and assume that m_p is incomparably greater than m_e ($m_p \gg m_e$).

Solution: (a) The potential energy of the electron is

$$V = -\frac{e^2}{4\pi\epsilon_o r},$$

and the kinetic energy

$$K = \frac{1}{2} m_e v^2.$$

Since the Coloumb force keeps the electron in a centripetal orbit,

$$\frac{1}{2} m_e v^2 = \frac{1}{2} \frac{e^2}{4\pi\epsilon_o r}.$$

Therefore,

$$\begin{aligned} E_{Total} = V + K &= -\frac{e^2}{4\pi\epsilon_o r} + \frac{e^2}{8\pi\epsilon_o r} \\ &= -\frac{e^2}{8\pi\epsilon_o r}. \end{aligned}$$

The orbital momentum is

$$L = m_e v r.$$

(b) From above, we have

$$m_e v^2 = \frac{e^2}{4\pi\epsilon_o r,}$$

Combining with the expression

$$L = m_e v r,$$

we obtain

$$r = \frac{4\pi\epsilon_o L^2}{e^2 m_e}.$$

Therefore,

$$E = -\frac{e^2}{8\pi\epsilon_o}\frac{e^2 m_e}{4\pi\epsilon_o L^2}$$

$$= -\frac{e^4 m_e}{32\pi^2 {\epsilon_o}^2 L^2}.$$