3 Medium Questions

1. **(30 points)** *The speed of light*

   The year is 1671 and you are astronomer Ole Rømer, measuring the period of Io's orbit around Jupiter by timing the passages of Io into or out of Jupiter's shadow.

   In December of 1671, Jupiter is at its first quadrature and you observe eclipses at the following times:

   - December 18 at 06:17:48
   - December 20 at 00:46:09
   - December 21 at 19:14:30

   In June of 1672, Jupiter is at its second quadrature and you observe eclipses at the following times:

   - June 19 at 08:42:50
   - June 21 at 03:11:30
   - June 22 at 21:40:10

   (a) **(2 points)** What is the interval between eclipses of Io as measured in December 1671, and what is it in June 1672?

   (b) **(7 points)** These orbital periods are slightly different. Rømer hypothesized that this is evidence that light has a finite speed. Explain why he thought this.

   (c) **(15 points)** Calculate the speed of light from these observations, and what you know about the orbits of Earth and Jupiter. Explain any simplifying assumptions that you make. How close is this speed of light to the correct value? (Hint: at second quadrature Earth is moving directly away from Jupiter, and at first quadrature Earth is moving directly towards Jupiter.)

   (d) **(6 points)** In 1672, Rømer did not have an accurate measurement of the distance from the Earth to the Sun. Write the speed of light *as he would have had to write it*, in terms of the unknown Earth-Sun distance a.

   (Be careful: Rømer also did not know the gravitational constant or the mass of the sun!)

Solution:

(a) In December 1671, the observed interval between successive eclipses is 42 hours, 28 minutes, and 21 seconds.

In June 1672, the observed interval between successive eclipses is 42 hours, 28 minutes, and 40 seconds.

Notice that the periods differ by 19 seconds.

(b) Since light has a finite speed, we don't observe the eclipses of Io until some time after they actually happen: there is a light delay time. When the distance between Jupiter and Earth is decreasing, the light delay time should be decreasing with each eclipse, so the time between eclipses seems to be shorter. Similarly, when the distance between Jupiter and Earth is increasing, the time between eclipses appears to be longer.

This is very similar to the modern concept of the Doppler effect, though Rømer didn't call it that at the time.

(c) Let the true orbital period be p, and the radial velocity of Jupiter relative to Earth be v. If the speed of light is c, then the delay time between successive orbits will change by $\frac{pv}{c}$. Therefore the observed orbital period is longer than the true orbital period by $\Delta p = \frac{pv}{c}$. The speed of light is thus

$$c = \frac{pv}{\Delta p}$$

Jupiter's orbital velocity is somewhat slower than Earth's, and it is also almost entirely tangential rather than radial. So we make the simplifying assumption that the radial velocity of Jupiter relative to Earth at first and second quadrature is just the Earth's orbital velocity (towards Earth at first quadrature and away from Earth at second quadrature.) So v is just the Earth's orbital velocity, which from Kepler's third law is

$$v = \sqrt{\frac{GM_{\text{sun}}}{a}} = 30 \text{ km/s}$$

The true orbital period of Io is roughly the average of the two measured periods, so $p =$ 42:28:30.5, and $\Delta p = 9.5$ sec. Plugging in these values to the above expression for the speed of light gives

$$c = 4.8 \cdot 10^8 \text{ m/s}$$

Today we know that this is about 60% larger than the true value, which is not bad given all the approximations that were made.

(d) We could try to carry out the same derivation as in the previous part, but leave a as an unknown constant. This gives

$$c = \frac{p\sqrt{GM_{\text{sun}}}}{\Delta p} a^{-1/2}$$

But this expression depends on G and M_{sun}, which Rømer would not have known at the time. Rømer did not know any of a, G and M_{sun} individually, but he did know that the Earth's orbital period

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM_{\text{sun}}}}$$

is one year.

So we can instead write

$$v = \frac{2\pi a}{1 \text{ year}}$$

$$c = \frac{2\pi p}{(1 \text{ year})\Delta p} a = \frac{a}{5\text{m}12\text{s}}$$

This is equivalent to saying that it takes 5 minutes and 12 seconds for light to travel from the Sun to the Earth. It can be argued that Rømer was not really measuring the speed of light, but instead just the light travel time from the Earth to the Sun. (The first accurate determination of the Earth-Sun distance was made a year later by Cassini.)

2. **(30 points)** A meteorite that is radially approaching the Earth collides with a space station that revolves around the Earth in a circular orbit with radius R. For all parts of the question, express your results in terms of the mass M of the Earth, the gravitational constant G, the mass m_1 of the meteorite, and the mass m_2 of the space station.

   (a) Assume that, after the impact, the meteorite and the space station form a conglomerate that moves in a closed orbit which approaches the center of the Earth at a minimum distance $R/2$. State what the shape of the orbit of the conglomerate is and determine:

(i) the speed of the meteorite just before the collision,
(ii) the minimum and maximum speeds of the conglomerate,
(iii) the maximum distance of the conglomerate from the center of the Earth.

Determine the condition that m_1 and m_2 must obey so that the aforementioned scenario is possible.

(b) Determine the minimum speed that the meteorite should have just before the collision so that the conglomerate moves in an open orbit after the impact. For this minimum value of the speed of the meteorite, state what the shape of the orbit of the conglomerate would be and determine

(i) the maximum speed of the conglomerate,
(ii) its minimum distance from the center of the Earth,
(iii) the angle traversed by the orbital radius from the moment of the collision until the moment when the conglomerate approaches the center of the Earth to minimum distance.

Solution:

(a) The orbit of the conglomerate after the collision will be an **ellipse**.
Conservation of momentum implies that the radial and tangential components (v_r and v_t, respectively) of the speed of the conglomerate are:

$$v_r = \frac{m_1}{m_1 + m_2} v$$

and

$$v_t = \frac{m_2}{m_1 + m_2} \sqrt{\frac{GM}{R}}.$$

Here, v is the speed of the meteorite prior to the collision.
Conservation of angular momentum of the conglomerate during its orbital motion implies

$$\frac{m_2}{m_1 + m_2} \sqrt{\frac{GM}{R}} R = v_{max} \frac{R}{2}.$$

Therefore, we deduce that

$$\boxed{v_{max} = \frac{2m_2}{m_1 + m_2} \sqrt{\frac{GM}{R}}}.$$

Finally, conservation of mechanical energy per unit mass of the conglomerate yields

$$\frac{1}{2} v_{max}^2 - \frac{GM}{R/2} = \frac{1}{2}(v_r^2 + v_t^2) - \frac{GM}{R}.$$

Combining all previous equations, we find

$$\left(\frac{2m_2}{m_1 + m_2}\right)^2 \frac{GM}{R} - \frac{4GM}{R} = \left(\frac{m_1}{m_1 + m_2}\right)^2 v^2 + \left(\frac{m_2}{m_1 + m_2}\right)^2 \frac{GM}{R} - \frac{2GM}{R},$$

which immediately yields

$$\boxed{v = \frac{\sqrt{3m_2^2 - 2(m_1 + m_2)^2}}{m_1} \sqrt{\frac{GM}{R}}},$$

provided that

$$\sqrt{3} m_2 > \sqrt{2}(m_1 + m_2) \Leftrightarrow \boxed{m_1 < \left(\sqrt{\frac{3}{2}} - 1\right) m_2}.$$

Now, one can find the semi-major axis of the orbit of the conglomerate:

$$\frac{1}{a} = \frac{2}{r_{min}} - \frac{v_{max}^2}{GM} = \frac{4}{R}\left(1 - \frac{m_2^2}{(m_1+m_2)^2}\right) \Leftrightarrow a = \frac{R}{4}\frac{(m_1+m_2)^2}{(m_1+m_2)^2 - m_2^2}.$$

Moreover, if we denote the eccentricity of the orbit by e, we find

$$a(1-e) = \frac{R}{2} \Leftrightarrow ae = a - \frac{R}{2}.$$

It follows that

$$r_{max} = a(1+e) = 2a - \frac{R}{2} \Leftrightarrow \boxed{r_{max} = \frac{R}{2}\frac{m_2^2}{(m_1+m_2)^2 - m_2^2}}.$$

Conservation of angular momentum immediately yields

$$v_{min} = v_{max}\frac{r_{min}}{r_{max}} \Leftrightarrow \boxed{v_{min} = \frac{2m_1(m_1+2m_2)}{m_2(m_1+m_2)}\sqrt{\frac{GM}{R}}}.$$

(b) The orbit of the conglomerate will have the shape of a **parabola**. This scenario will occur when

$$v_r^2 + v_t^2 = \frac{2GM}{R} \Leftrightarrow \boxed{v = \frac{\sqrt{2(m_1+m_2)^2 - m_2^2}}{m_1}\sqrt{\frac{GM}{R}}}.$$

Conservation of mechanical energy gives

$$v_{max}^2 = \frac{2GM}{r_{min}},$$

while conservation of angular momentum implies

$$v_{max} r_{min} = \frac{m_2}{m_1+m_2}\sqrt{\frac{GM}{R}}R.$$

Therefore,

$$\boxed{v_{max} = \frac{2(m_1+m_2)}{m_2}\sqrt{\frac{GM}{R}}}$$

and

$$r_{min} = \frac{m_2^2}{2(m_1+m_2)^2}R.$$

From the equation of a parabola in polar coordinates,

$$r = \frac{2r_{min}}{1+\cos\theta},$$

one can find

$$\frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2} = \frac{r_{min}}{r} \Leftrightarrow \theta = 2\arccos\sqrt{\frac{r_{min}}{R}} = 2\arccos\frac{m_2}{\sqrt{2}(m_1+m_2)}.$$