4 Long Questions

1. **(45 points)** *The Curious Orbit of James Webb*

   For his upcoming Astrophysics Club presentation, Will researches the recently launched James Webb Space Telescope (JWST), the next-generation telescope designed as the successor of the Hubble Space Telescope. The largest space telescope ever built, the JWST uses its large collecting area to observe in the infrared spectrum. It orbits around the L_2 Lagrange point of the Earth-Sun system. Lagrange points are equilibrium points for a small body in the Earth-Sun system; L_2 is the point on the Earth-Sun line located beyond Earth's orbit.

   In the problem, let M and m be the mass of the Sun and Earth, respectively, with $M \gg m$. Additionally, consider the Sun's and Earth's radius to be $R_\odot$ and $R_\oplus$ respectively, and the Earth to orbit the Sun in a perfectly circular orbit of radius R.

   (a) **(2 points)** The orbit of JWST was designed to circle around L_2 in a big enough orbit to avoid Earth's shadow. What is the benefit of i) being at a Lagrangian point and ii) avoiding Earth's shadow?

   (b) **(5 points)** Taking first order approximations, about how far is L_2 from Earth? Express your answer both in terms of the variables defined and numerically, in km.

   (c) **(5 points)** In the rotating reference frame in which the Earth and the Sun are stationary, JWST orbits L_2 in the plane perpendicular to the Earth-Sun line that passes through L_2. If JWST orbits in a circle of radius r around L_2 in this frame, what is the minimum r that avoids the Earth's shadow at all times? Express your answer both in terms of the variables defined and numerically, in km.

   (d) **(20 points)** Consider a scenario where JWST is stationary in the aforementioned rotating reference frame and has a small displacement $\boldsymbol{\delta r} = \delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}}$ relative to L_2, where $\hat{\boldsymbol{\imath}}$ is the unit vector along the Earth-Sun line away from the Sun and $\hat{\boldsymbol{\jmath}}$ is a unit vector perpendicular to $\hat{\boldsymbol{\imath}}$. Both $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are stationary in the rotating frame. To first order (i.e. assuming $|\boldsymbol{\delta r}| \ll x$), what is the acceleration of JWST in the rotating frame?

   (e) **(5 points)** The presence of the Coriolis force in the rotating reference frame destabilizes orbits around L_1, L_2, and L_3 while stabilizing orbits around L_4 and L_5. **Disregarding the Coriolis force for this part only,** are orbits stable around L_2 when there is no Coriolis force? Is this result generalizable? In other words, what can be said about the stability of orbits around an arbitrary, stationary point where there are no masses within the orbit and no fictitious forces involved?

   (f) **(8 points)** Suppose JWST orbits in the circle described in part (c) with a constant speed and an orbital radius of 500,000 km. Suppose further that the jet propulsion of the JWST is programmed to counteract only the Coriolis force; the rest of JWST's motion is due to the natural gravitational dynamics at L_2. Using the assumption that the first order expression derived in (d) still applies, if JWST has a mass of 6500 kg, what is the average magnitude of the force over a long period of time? The following averages (calculated from 0 to 2π) might be helpful:

   $$\overline{|\sin x|} = \frac{2}{\pi} \qquad \overline{\sin^2 x} = \frac{1}{2} \qquad \overline{|\sin^3 x|} = \frac{4}{3\pi}$$

   For reference, the magnitude of the Coriolis force is given as,

   $$|\boldsymbol{F}| = 2m|\boldsymbol{\omega} \times \boldsymbol{v}|$$

Solution:

(a) i) A Lagrangian point is stationary with respect to the Earth and the Sun, which is beneficial for a space telescope because it requires little fuel to maintain its position.

ii) Avoiding Earth's shadow is beneficial for JWST because it always has access to the Sun's energy, never to be interrupted by eclipses.

(b) This is a classic problem. L_2 is the Lagrangian point just beyond Earth's orbit. Let us define the L_2-Earth distance to be x, where $x \ll R$. Taking into account the forces of the Earth and the Sun yields:

$$\frac{GM}{(R+x)^2} + \frac{Gm}{x^2} = \omega^2(R+x)$$

where $\omega = \sqrt{GM/R^3}$ is the rotational velocity of Earth's orbit. Taking advantage of the binomial approximation $(1+y)^\alpha \approx 1 + \alpha y$ when $y \ll 1$, we can simplify our expression to the following:

$$\frac{GM}{R^2}\left(1 - \frac{2x}{R}\right) + \frac{Gm}{x^2} \approx \frac{GM}{R^2}\left(1 + \frac{x}{R}\right)$$

Solving the equation yields

$$\boxed{x = R\sqrt[3]{\frac{m}{3M}} = 1.50 \times 10^6 \text{ km}}$$

(c) From geometry, JWST must lie outside of the cone that is defined by the common internal tangents of the Earth and the Sun. The tip of the cone has an angle

$$2\theta = 2\sin^{-1}\left(\frac{R_\odot + R_\oplus}{R}\right)$$

If you extend one of those internal tangents as follows (insert diagram), one can see that r_{min} satisfies the following equation:

$$\tan\theta = \frac{R_\odot/\cos\theta + r}{R+x}$$

Therefore:

$$\boxed{r_{\text{min}} = \frac{(R+x)R_\oplus + xR_\odot}{\sqrt{R^2 - (R_\odot + R_\oplus)^2}} = 1.34 \times 10^4 \text{ km}}$$

Note that realistically, $R_\oplus, R_\odot \ll R$ and $x \ll R$, so $r_{\text{min}} \approx R_\oplus + xR_\odot/R$, which is off by less than 1%.

(d) Because L_2 is a Lagrangian point, we know that in the rotating frame, the forces at L_2 balance. Therefore, we only need to find first order changes in force relative to L_2. Suppose a mass M is located at a distance R to the left of L_2. Then the change in gravitational field from L_2 to $L_2 + \boldsymbol{\delta r}$ is

$$\boldsymbol{\delta g} = -\frac{GM}{((R+\delta x)^2 + \delta y^2)^{3/2}}((R+x)\boldsymbol{\hat{\imath}} + \delta y\boldsymbol{\hat{\jmath}}) + \frac{GM}{R^2}\boldsymbol{\hat{\imath}}$$

The δy^2 in the denominator of the first term is of second order and can be discarded:

$$\boldsymbol{\delta g} \approx -\left(\frac{GM}{(R+\delta x)^3}(R+x) - \frac{GM}{R^2}\right)\hat{\boldsymbol{\imath}} - \frac{GM}{R^3}\delta y\hat{\boldsymbol{\jmath}}$$

The binomial approximation can be used once again:

$$\boldsymbol{\delta g} \approx \frac{GM}{R^2}\left(1 - \frac{2x}{R} - 1\right)\hat{\boldsymbol{\imath}} - \frac{GM}{R^3}\delta y\hat{\boldsymbol{\jmath}} = \frac{GM}{R^3}(2\delta x\hat{\boldsymbol{\imath}} - \delta y\hat{\boldsymbol{\jmath}})$$

You might notice that this expression looks similar to that of tidal forces, because what we've done here is exactly the same! Now, to find the acceleration of JWST, we simply need to add the $\boldsymbol{\delta g}$ of the Earth and the Sun, then add $\boldsymbol{\delta a}$ of the centrifugal force on top (the coriolis force is 0 since JWST is stationary). $\boldsymbol{\delta a}$ due to the centrifugal force is simply:

$$\boldsymbol{\delta a} = \delta(\omega^2\boldsymbol{r}) = \omega^2\boldsymbol{\delta r} = \frac{GM}{R^3}(\delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}})$$

Therefore, the acceleration of JWST is

$$\boldsymbol{a} = \left(\frac{GM}{(R+x)^3} + \frac{Gm}{x^3}\right)(2\delta x\hat{\boldsymbol{\imath}} - \delta y\hat{\boldsymbol{\jmath}}) + \frac{GM}{R^3}(\delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}})$$

Plugging in $x = R\sqrt[3]{m/3M}$ and sticking to first order in $\boldsymbol{\delta r}$ yields

$$\begin{aligned}\boldsymbol{a} &\approx \left(2\left(\frac{GM}{R^3} + \frac{3GM}{R^3}\right) + \frac{GM}{R^3}\right)\delta x\hat{\boldsymbol{\imath}} + \left(-\left(\frac{GM}{R^3} + \frac{3GM}{R^3}\right) + \frac{GM}{R^3}\right)\delta y\hat{\boldsymbol{\jmath}} \\ &= \boxed{\frac{GM}{R^3}(9\delta x\hat{\boldsymbol{\imath}} - 3\delta y\hat{\boldsymbol{\jmath}})}\end{aligned}$$

(e) From our expression in part (d), we see that a small displacement in the $\hat{\boldsymbol{\imath}}$ direction results in a force that points in the same direction, meaning any orbits are $\boxed{\text{unstable}}$ in that direction.

In fact, this result is general for all orbits around arbitrary points with no mass and no fictitious forces. Using Gauss's law for gravitation, the flux through a surface surrounding such a point must be 0, indicating it is impossible for the gravitational force to be pointing inward for displacements in every possible direction.

(f) The acceleration in the direction perpendicular to $\hat{\boldsymbol{\imath}}$ was found in part (d) to be $-3GM\delta y/R^3$. Setting this equal to the centripetal acceleration, we have

$$\begin{aligned}\frac{3GMr}{R^3} &= \frac{v^2}{r} \\ v &= r\sqrt{\frac{3GM}{R^3}}\end{aligned}$$

The magnitude of the Coriolis force is

$$|\boldsymbol{F}| = 2m|\boldsymbol{\omega} \times \boldsymbol{v}| = 2m\sqrt{\frac{GM}{R^3}}r\sqrt{\frac{3GM}{R^3}}|\sin\theta|$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{\omega}$. Since it varies from 0 to 2π, we use the given value of $\overline{|\sin\theta|}$ to find the average magnitude to be

$$\boxed{|\boldsymbol{F}| = \frac{4\sqrt{3}GMmr}{\pi R^3} = 0.28\ \mathrm{N}}$$

Note just how small the force is!

2. **(45 points)** *The Sundial I*

While on a walk in Princeton University, Leo stumbled upon the following sundial, mounted on the southern wall of a building:

Figure 1: Picture taken by Leo Yao, December 2020.

He was familiar with the lines pointing outwards from the center, marking off time of day. However, he also noticed the three curves crossing the other lines. After a bit of thought, he realized these curves marked off the path of the shadow on the equinoxes and solstices.

(a) **(3 points)** For each of the two equinoxes and solstices, match the day to the curve (top, middle, bottom) denoting the path of the shadow on that day. (No explanation needed)

(b) **(1 point)** For the days corresponding to the middle curve, what is the declination of the Sun on those days? Assume the length of the day is small compared to the length of the year. (No explanation needed)

He then noticed that the middle curve seemed to be a straight line, and started thinking about if this is the case. He first considered a simpler system: a stick mounted vertically on a flat surface, casting a shadow on flat ground.

(c) **(9 points)** Consider the shadow of the tip of the stick, which might possibly trace a straight line over the course of the day. Explain why, if this happens, it can only happen on a day when the Sun's declination is that determined above.

This part can be solved independently, or as part of your solution for the next part. If your solution for the next part also proves this, note that down on your solution sheets, and proceed directly to the next part.

(d) **(24 points)** Prove that, for the Sun's declination determined above, the shadow of the tip of the stick traces a straight line over the course of the day.

Any method is acceptable, as long as it is presented clearly and rigorously. For example, one possible method might involve the following steps:

i) Determining the orientation of the line and explaining why it must be in this orientation;
ii) Determining the length of the shadow for a given position of the Sun in alt-az coordinates;
iii) Deriving a relation between altitude and azimuth given that the tip of the shadow is on the line;
iv) Determining a constant quantity and showing that it is constant over all positions of the Sun that day.

If you skipped the previous part, make sure your proof also shows the inverse: that for a different declination of the Sun, the tip's shadow does not trace a straight line.

You do not necessarily need to follow these steps. Simpler and/or faster methods may be possible, including those that do not need any equations. **Any fully-formed, valid explanation gives full credit.**

After figuring out the simpler case, Leo realized that he could easily generalize it to the sundial mounted on the wall. He then started thinking about other ways the model and the sundial on the wall differed, and thought about the orientation of the center line, noticing that it was not perfectly horizontal, but instead slanted.

(e) **(8 points)** Based on the picture and the orientation of the center line, does the wall run perfectly East-West, or does it run Northeast-Southwest or Southeast-Northwest? Explanation needed for credit.

Assume the wall is perfectly vertical. **This part can be solved independently of the previous two parts.**

Solution:

(a) The Sun is highest on the summer solstice, and lowest on the winter solstice. Therefore shadows for sundials on the ground are shorter in summer than in winter. However, this sundial is on a vertical wall, so this trend is actually reversed: the top curve (shortest shadows) corresponds to the winter solstice and the bottom curve (longest shadows) corresponds to the summer solstice.

Also notice the visible shadow close to the top curve; as the picture was taken in December 2020, this shadow corresponds to the winter solstice and again the top curve.

The equinoxes therefore correspond to the curve in the middle.

(b) On the equinoxes, the declination of the Sun is $\delta = 0$.

(c) There are many possible solutions to this part; here we present one.

First note that, as the Sun's path is symmetric across the meridian, by symmetry, the line must run perfectly East to West.

Consider the Sun's position at solar noon on any day; if the Sun does not rise, no line is possible. But if the Sun does rise, the altitude of the Sun as it crosses the meridian is nonzero. For a finite stick height, the shadow length is finite. Therefore, the line must be a finite distance from the stick.

Now consider the azimuth of the Sun at sunset. If $\alpha \neq 270$, then there is some component of the shadow that runs North-South. But as the Sun's altitude goes to 0, the shadow length goes to ∞. As the line runs East-West, the line must therefore be an infinite distance from the stick, which is a contradiction.

Therefore, this is only possible for $\alpha = 270$ at sunset, implying $\delta = 0$.

(d) We present a variety of solutions. First we present the method described in the problem statement:

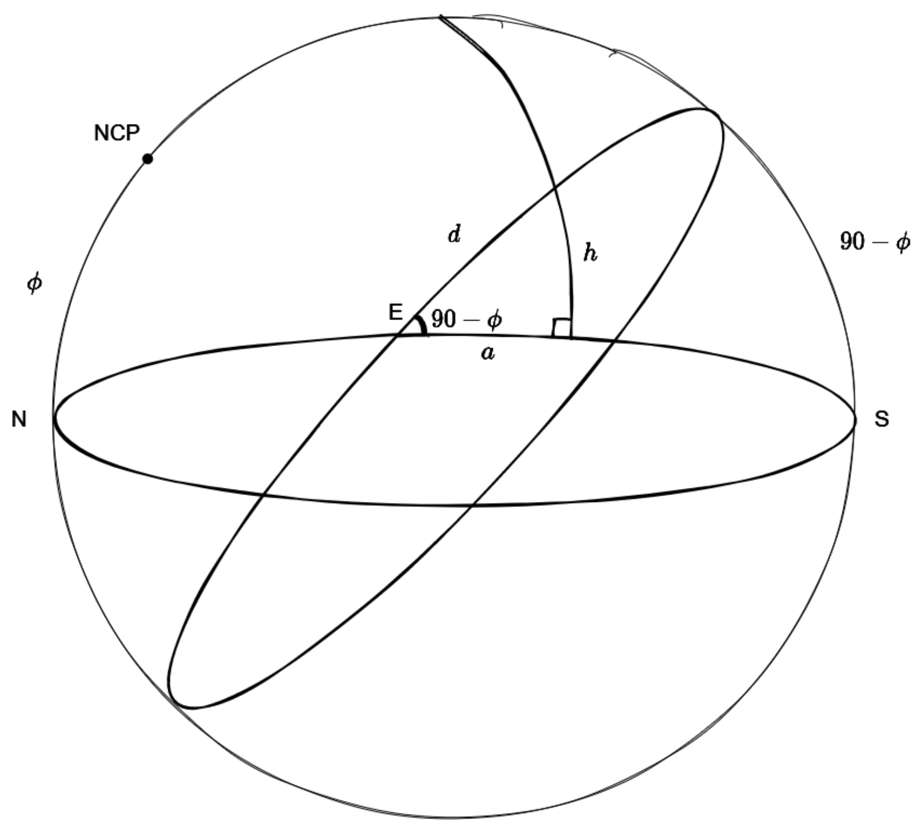
i) From the previous part, we know the line must run perfectly East to West, at a finite distance from the stick.

ii) If the Sun has altitude h, then if the stick has length h_0, then the length of the shadow is given by $l = \frac{h_0}{\tan h}$.

iii) Let $a = \text{azimuth} - 90$, such that the Sun rises at $a = 0$ and sets at $a = 180$. Let the line be a distance x_0 from the stick; then we need $l \sin a = x_0$.

iv) Therefore, $\frac{h_0 \sin a}{\tan h} = x_0$, which we need to hold for all positions (a, h) of the Sun over the entire day. We have therefore derived the constant quantity $\frac{\sin a}{\tan h}$.
To finish off the proof, we need to show this is a constant quantity:

$$\frac{\sin a}{\tan h} = \frac{\sin a \cos h}{\sin h}$$



Using the spherical law of cosines:

$$= \frac{\sin a}{\sin h} \left(\cos a \cos d + \sin a \sin d \cos(90 - \phi)\right)$$

Using the spherical right triangle equality $\cos d = \cos h \cos a$:

$$= \frac{\sin a}{\sin h}\left(\cos^2 a \cos h + \sin a \sin d \sin\phi\right)$$

$$= \cos^2 a \frac{\sin a \cos h}{\sin h} + \sin^2 a \frac{\sin d}{\sin h}\sin\phi$$

Noticing the first term has the same form as the original:

$$(1 - \cos^2 a)\frac{\sin a}{\tan h} = \sin^2 a \frac{\sin d}{\sin h}\sin\phi$$

Using $1 - \cos^2 a = \sin^2 a$ and cancelling terms:

$$\frac{\sin a}{\tan h} = \frac{\sin d}{\sin h}\sin\phi$$

Using the spherical law of sines $\frac{\sin h}{\sin(90-\phi)} = \sin d$:

$$= \frac{\sin\phi}{\sin(90-\phi)} = \tan\phi$$

Therefore, this is a constant quantity, which means the tip of the shadow is always on this straight line.

If the previous part was skipped, it is also necessary to prove the line is perfectly East to West, and that it is a finite distance from the stick, as those assumptions were used in the derivation above.

The inverse would also need to be correctly justified. For example, one way to do this would be to simplify $\frac{\sin a}{\tan h}$ without the $\delta = 0$ assumption.

Using the spherical law of cosines to derive the coordinate transformation:

$$\cos(90-\delta) = \cos(90-\phi)\cos(90-h) + \sin(90-\phi)\sin(90-h)\cos(a+90)$$

$$\sin\delta = \sin\phi\sin h - \cos\phi\cos h\sin a$$

$$\sin a = \frac{\sin\phi\sin h - \sin\delta}{\cos\phi\cos h}$$

Applying this transformation:

$$\frac{\sin a}{\tan h} = \frac{1}{\tan h}\frac{\sin\phi\sin h - \sin\delta}{\cos\phi\cos h}$$

$$= \frac{\sin\phi\sin h}{\cos\phi\sin h} - \frac{\sin\delta}{\sin h\cos\phi}$$

$$= \tan\phi - \frac{\sin\delta}{\cos\phi}\frac{1}{\sin h}$$

The latitude ϕ is a constant, and the declination δ is approximately unchanged over the course of a day. However, $\frac{1}{\sin h}$ varies over the course of a day as the Sun rises and sets. Therefore, for this to be a conserved quantity, we need $\sin\delta = 0$, implying $\delta = 0$.

However, this problem can be solved much faster geometrically. A simple solution using nothing but properties of great circles and planes goes as follows:

Place the tip of the stick at the center of the celestial sphere. On the equinoxes, the declination of the Sun is 0, so the path of the Sun is a great circle; therefore the path of the Sun through the day and the tip of the stick lie in the same plane. Therefore, the shadow of the tip of the stick must be in the same plane. Intersecting this plane with the plane of the ground gives a straight line for the shadow's path. □

Notice that this solution quickly generalizes to any other intersecting plane, be it a vertical wall or any other orientation the sundial is mounted in.

(e) Consider two positions of the Sun on the equinox, equidistant from noon. By symmetry, the altitudes are the same, and the aziumths are equally offset from due South.

Consider the shadow cast by the Sun at each of these times, which is the intersection of the line connecting the Sun's position and the tip of the sundial with the wall. As the Sun is up in the South, these lines slope down as they head North.

Looking at the line in the picture, the line traveling West drops less compared to the line traveling East. This implies the wall juts out towards the observers perspective on the left, meaning the wall is angled Southwest to Northeast.

Looking at the actual building, Fisher Hall, on Google Maps, it indeed is oriented that way!

Problem adapted and extended from Roberto Bozcko and Lucas Carrit Delgado Pinheiro.