4 Long Questions

1. **(45 points)** *The Curious Orbit of James Webb*

   For his upcoming Astrophysics Club presentation, Will researches the recently launched James Webb Space Telescope (JWST), the next-generation telescope designed as the successor of the Hubble Space Telescope. The largest space telescope ever built, the JWST uses its large collecting area to observe in the infrared spectrum. It orbits around the L_2 Lagrange point of the Earth-Sun system. Lagrange points are equilibrium points for a small body in the Earth-Sun system; L_2 is the point on the Earth-Sun line located beyond Earth's orbit.

   In the problem, let M and m be the mass of the Sun and Earth, respectively, with $M \gg m$. Additionally, consider the Sun's and Earth's radius to be $R_\odot$ and $R_\oplus$ respectively, and the Earth to orbit the Sun in a perfectly circular orbit of radius R.

   (a) **(2 points)** The orbit of JWST was designed to circle around L_2 in a big enough orbit to avoid Earth's shadow. What is the benefit of i) being at a Lagrangian point and ii) avoiding Earth's shadow?

   (b) **(5 points)** Taking first order approximations, about how far is L_2 from Earth? Express your answer both in terms of the variables defined and numerically, in km.

   (c) **(5 points)** In the rotating reference frame in which the Earth and the Sun are stationary, JWST orbits L_2 in the plane perpendicular to the Earth-Sun line that passes through L_2. If JWST orbits in a circle of radius r around L_2 in this frame, what is the minimum r that avoids the Earth's shadow at all times? Express your answer both in terms of the variables defined and numerically, in km.

   (d) **(20 points)** Consider a scenario where JWST is stationary in the aforementioned rotating reference frame and has a small displacement $\boldsymbol{\delta r} = \delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}}$ relative to L_2, where $\hat{\boldsymbol{\imath}}$ is the unit vector along the Earth-Sun line away from the Sun and $\hat{\boldsymbol{\jmath}}$ is a unit vector perpendicular to $\hat{\boldsymbol{\imath}}$. Both $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are stationary in the rotating frame. To first order (i.e. assuming $|\boldsymbol{\delta r}| \ll x$), what is the acceleration of JWST in the rotating frame?

   (e) **(5 points)** The presence of the Coriolis force in the rotating reference frame destabilizes orbits around L_1, L_2, and L_3 while stabilizing orbits around L_4 and L_5. **Disregarding the Coriolis force for this part only,** are orbits stable around L_2 when there is no Coriolis force? Is this result generalizable? In other words, what can be said about the stability of orbits around an arbitrary, stationary point where there are no masses within the orbit and no fictitious forces involved?

   (f) **(8 points)** Suppose JWST orbits in the circle described in part (c) with a constant speed and an orbital radius of 500,000 km. Suppose further that the jet propulsion of the JWST is programmed to counteract only the Coriolis force; the rest of JWST's motion is due to the natural gravitational dynamics at L_2. Using the assumption that the first order expression derived in (d) still applies, if JWST has a mass of 6500 kg, what is the average magnitude of the force over a long period of time? The following averages (calculated from 0 to 2π) might be helpful:

   $$\overline{|\sin x|} = \frac{2}{\pi} \qquad \overline{\sin^2 x} = \frac{1}{2} \qquad \overline{|\sin^3 x|} = \frac{4}{3\pi}$$

   For reference, the magnitude of the Coriolis force is given as,

   $$|\boldsymbol{F}| = 2m|\boldsymbol{\omega} \times \boldsymbol{v}|$$

2. **(45 points)** *The Sundial I*

   While on a walk in Princeton University, Leo stumbled upon the following sundial, mounted on the southern wall of a building:

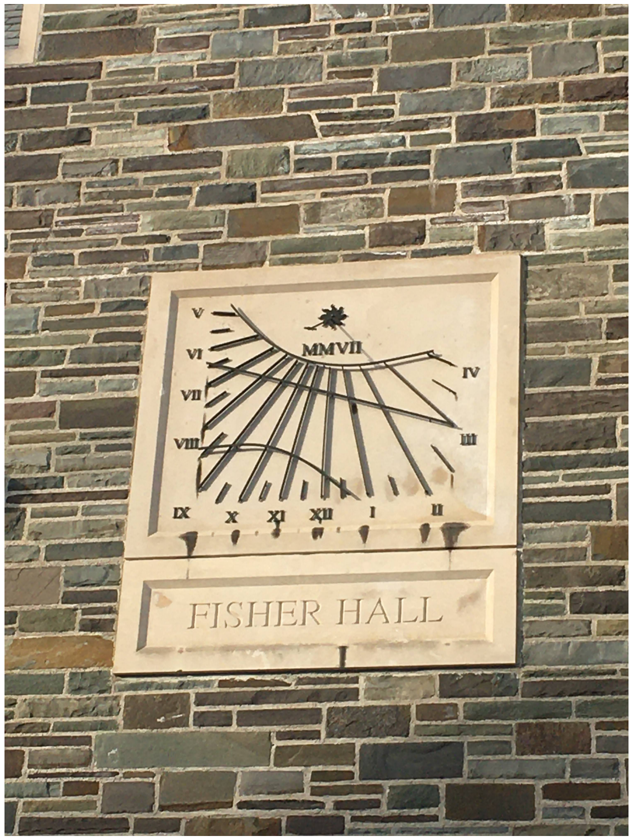

Figure 1: Picture taken by Leo Yao, December 2020.

He was familiar with the lines pointing outwards from the center, marking off time of day. However, he also noticed the three curves crossing the other lines. After a bit of thought, he realized these curves marked off the path of the shadow on the equinoxes and solstices.

(a) **(3 points)** For each of the two equinoxes and solstices, match the day to the curve (top, middle, bottom) denoting the path of the shadow on that day. (No explanation needed)

(b) **(1 point)** For the days corresponding to the middle curve, what is the declination of the Sun on those days? Assume the length of the day is small compared to the length of the year. (No explanation needed)

He then noticed that the middle curve seemed to be a straight line, and started thinking about if this is the case. He first considered a simpler system: a stick mounted vertically on a flat surface, casting a shadow on flat ground.

(c) **(9 points)** Consider the shadow of the tip of the stick, which might possibly trace a straight line over the course of the day. Explain why, if this happens, it can only happen on a day when the Sun's declination is that determined above.

This part can be solved independently, or as part of your solution for the next part. If your solution for the next part also proves this, note that down on your solution sheets, and proceed directly to the next part.

(d) **(24 points)** Prove that, for the Sun's declination determined above, the shadow of the tip of the stick traces a straight line over the course of the day.

Any method is acceptable, as long as it is presented clearly and rigorously. For example, one possible method might involve the following steps:

i) Determining the orientation of the line and explaining why it must be in this orientation;
ii) Determining the length of the shadow for a given position of the Sun in alt-az coordinates;
iii) Deriving a relation between altitude and azimuth given that the tip of the shadow is on the line;
iv) Determining a constant quantity and showing that it is constant over all positions of the Sun that day.

If you skipped the previous part, make sure your proof also shows the inverse: that for a different declination of the Sun, the tip's shadow does not trace a straight line.

You do not necessarily need to follow these steps. Simpler and/or faster methods may be possible, including those that do not need any equations. **Any fully-formed, valid explanation gives full credit.**

After figuring out the simpler case, Leo realized that he could easily generalize it to the sundial mounted on the wall. He then started thinking about other ways the model and the sundial on the wall differed, and thought about the orientation of the center line, noticing that it was not perfectly horizontal, but instead slanted.

(e) **(8 points)** Based on the picture and the orientation of the center line, does the wall run perfectly East-West, or does it run Northeast-Southwest or Southeast-Northwest? Explanation needed for credit.

Assume the wall is perfectly vertical. **This part can be solved independently of the previous two parts.**