USAAAO 2021 - First Round

January 30^{th}, 2021

PHYSICAL AND ASTRONOMICAL CONSTANTS

c	Speed of light in vacuum	2.998×10^8 m s^{-1}
e	Elementary charge	1.602×10^{-19} C
m_n	Neutron rest mass	1.675×10^{-27} kg
m_p	Proton rest mass	1.6725×10^{-27} kg
m_e	Electron rest mass	9.110×10^{-31} kg
m_{He}	Helium-4 rest mass	6.644×10^{-27} kg
h	Planck's constant	6.626×10^{-34} J s
H_0	Hubble's constant	70 (km/s)/Mpc
k_B	Boltzmann's constant	1.381×10^{-23} J K^{-1}
b	Wien's constant	2.898×10^{-3} m K
G	Gravitational constant	6.673×10^{-11} N m^2 kg^{-2}
σ	Stefan-Boltzmann constant	5.670×10^{-8} J m^{-2} K^{-4} s^{-1}
c_1	First Radiation Constant $(= 2\pi hc^2)$	3.742×10^{-16} J m^2 s^{-1}
c_2	Second Radiation Constant $(= hc/k)$	1.439×10^{-2} m K
N_A	Avogadro constant	6.022×10^{23} mol^{-1}
R	Gas constant	8.314 J K^{-1} mol^{-1}
a_0	Bohr radius	5.292×10^{-11} m
μ_B	Bohr magneton	9.274×10^{-24} J T^{-1}
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.96×10^8 m
$L_\odot$	Solar luminosity	3.827×10^{26} J s^{-1}
$T_\odot$	Solar temperature	5770 K
$M_\oplus$	Earth mass	5.976×10^{24} kg
$R_\oplus$	Mean Earth radius	6.371×10^6 m
$I_\oplus$	Earth moment of Inertia	8.04×10^{37} kg m^2
$R_☾$	Mean Moon radius	1.737×10^6 m
$M_♃$	Mean Jupiter mass	1.9×10^{27} kg
$R_♃$	Mean Jupiter radius	7.1492×10^7 m
$a_♃$	Mean orbital radius of Jupiter	5.2 AU
$a_☾$	Mean semimajor axis Moon orbit	3.84399×10^8 m
1 light year		9.461×10^{15} m
1 AU	Astronomical Unit	1.496×10^{11} m
1 pc	Parsec	3.086×10^{16} m
1 year		3.156×10^7 s
1 sidereal day		86164 s
1 erg		1×10^{-7} J
1 bar		10^5 N m^{-2}

ENERGY CONVERSION : 1 joule (J) = 6.2415×10^{18} electronvolts (eV)

1. On December 21, 2020, Jupiter was at $(\alpha, \delta) = (20^h\,10^m, -20^\circ\,34')$. Which constellation was Saturn in?
   (a) Capricornus
   (b) Aquarius
   (c) Pisces
   (d) Aquila

Solution:

December 21, 2020 is the date of the Great Conjunction; thus Saturn and Jupiter are at the same coordinates in the same constellation. Since Saturn and Jupiter are solar system objections, they must be in a Zodiac constellation on the ecliptic. The vernal equinox is in Pisces, near its Western border. Approximating the zodiac constellations as chunks of 2 hours of Right Ascension, we can use the order of the Zodiac constellations to find that **Capricornus** spans approximately RA $20^h - 22^h$. Saturn and Jupiter were very close to the Capricornus-Sagittarius border, but Sagittarius was not included as an answer choice.

Answer: A

2. What is the spectral type of a star with a luminosity of $5.86 * 10^{26}$ W and radius of $8.51 * 10^8$ m?
   (a) A
   (b) F
   (c) G
   (d) K
   (e) M

Solution:

Using Stefan-Boltzmann's Law:

$$L = 4 * \pi * R^2 * \sigma * T^4$$

$$5.86 * 10^{26} = 4 * \pi * (8.51 * 10^8)^2 * 5.67 * 10^{-8} * T^4$$

$$T = 5805K$$

This is extremely close to the temperature of the sun, so the spectral type must be G.

Answer: C

3. The exoplanet HD 209458b has a mass of 0.71 Jupiter masses and orbits HD 209458 with an orbital period of 3.53 days. HD 209458 has a mass of 1.15 Solar masses. Assuming that the orbit

of HD 209458b is circular (which is a good approximation here) and that its orbit lies perfectly in our line of sight, what is the radial velocity semi-amplitude of HD 209458 due to the orbital motion of HD 209458b, in m/s?

(a) 69.6 m/s
(b) 85.9 m/s
(c) 94.2 m/s
(d) 120.8 m/s

Solution:

Using conservation of angular momentum, $M_p/M_s = V_s/V_p$. Assuming a circular orbit and solving for semi-major axis using Kepler's 3rd law, $V_p = 2\pi a/T = 2\pi \times 7.13 \times 10^9 m/(3.53 days \times 86400 s/day) = 146.8 km/s$. Plugging into the expression above, we find $V_s = 0.71 M_{Jup} \times 1.898 \times 10^{27} kg/(1.15 M_{Sun} \times 2 \times 10^{30} kg) \times 146.8 \times 10^3 m/s = 85.9 m/s$.

Answer: B

4. The photon number density of a blackbody depends on temperature as $n_d = a\left(\frac{k_B T}{\hbar c}\right)^n$ where k_B is the Boltzman constant, $\hbar$ is the reduced Planck's constant, c is the speed of light and T is the blackbody temperature. Here, a is a dimensionless numerical constant. What is the value of n?

(a) 1
(b) 2
(c) 3
(d) 4

Solution:

Use dimensional analysis.

Answer: C

5. HD 209458b has a radius of 1.35 Jupiter radii, while the radius of HD 209458 is 1.20 Solar radii. What is the transit depth of HD 209458b, in percent?

(a) 0.013%
(b) 0.13%
(c) 1.3%
(d) 13%

Solution:

Transit depth $= (R_p/R_s)^2$. In percent, transit depth $= 100(1.35 \times 69.9 \times 10^6 m/(1.20 \times 6.957 \times 10^8 m))^2 \approx 1.3\%$

Answer: C

6. Which of the following is a problem of the conventional Big Bang theory that is resolved by the theory of inflation?
   (a) Under the conventional Big Bang theory, it is extremely unlikely for our universe to be flat or nearly flat today, contrary to observation.
   (b) Under the conventional Big Bang theory, it is impossible for the Cosmic Microwave Background to have come into thermal equilibrium by the time of recombination, despite its observed uniform temperature.
   (c) The conventional Big Bang theory predicts a huge abundance of magnetic monopoles, while no magnetic monopoles have ever been discovered.
   (d) All of the above

Solution:

Each of (a), (b), and (c) are correct. Answer choice (a) is known as the flatness problem. As the universe expands, any deviation from flatness $|\Omega - 1|$ will grow quickly with time, according to the Friedmann equation. Thus, in order to have the flat ($|\Omega - 1| < 0.01$) universe that we observe today, the early universe must have been extremely flat (down to 18 decimal places at $t = 1$ second), which is a bizarrely fine-tuned initial condition. Inflation would rapidly flatten the universe, resolving this problem. Answer choice (b) is known as the horizon problem. The Cosmic Microwave Background is observed today to be very uniform in all directions, implying that the universe was in thermal equilibrium at recombination (when CMB radiation was released). However, the conventional Big Bang theory predicts the horizon distance was much smaller than the surface of last scattering (i.e. that light would not have had enough time to travel between opposite points of the CMB), so such a thermal equilibrium should be impossible. Inflation resolves this, since it is possible for the universe to be in thermal equilibrium before inflation. Then, the rapid expansion from inflation would create a universe uniform in temperature regardless of the horizon distance at recombination. Answer choice (c) is the magnetic monopole problem. Grand Unified Theories predict the existence of magnetic monopoles in the very early universe (perhaps 1 in every cubic horizon distance). Not only has no one discovered any magnetic monopoles, the conventional Big Bang theory actually predicts an abundance of magnetic monopoles so large that the density parameter Ω would exceed 10^{20}! If the magnetic monopoles formed before inflation, however, the density of magnetic monopoles would drop exponentially, making them impossible to detect today. These three answer choices are the three primary problems of the conventional Big Bang theory, leading to the cosmologists to supplement it with the theory of inflation.

Answer: D

7. Comet C/2020 F3 (NEOWISE) last reached perihelion on July 3, 2020. Comet NEOWISE has an orbital period of ≈ 4400 years and its eccentricity is 0.99921. What is the perihelion distance of Comet NEOWISE, in AU?
   (a) 0.0123 AU
   (b) 0.212 AU
   (c) 2.69 AU
   (d) 26.8 AU

Solution:

Perihelion $= a(1-e) = (M_{\text{Sun}}P^2)^{1/3}(1-e) = (1 \times 4400^2)^{1/3}(1-0.99921) = 0.212$ AU.

Answer: B

8. An astronomer detected a galaxy and decided to analyze its different parts and physical aspects. The frequency generated by a "spin-flip" transition of atomic hydrogen is $\nu_0 = 1420.406MHz$, however it was detected on the galaxy as $\nu = 1422.73$. He finds that:

   1. Population I stars are (1) and are metal-(2).
   2. The galaxy is (3) from us with a speed of (4) $km * s^{-1}$.

   Choose the alternative that correctly completes sentences above.

   (a) (1) young; (2) poor; (3) distancing; (4) 245
   (b) (1) old; (2) rich; (3) approaching; (4) 490
   (c) (1)old; (2) poor; (3) distancing; (4) 490
   (d) (1) young; (2) rich; (3) approaching; (4) 490
   (e) (1) young; (2) rich; (3) approaching; (4) 245

Solution:

By definition, population I stars consist of young metal-rich stars.

The detected frequency is higher than the laboratory one, which means that the galaxy is approaching us. Lets calculate this speed.

$$\nu = \nu_0\sqrt{\frac{c+v}{c-v}}$$

$$\frac{v}{c} = \frac{(\frac{\nu}{\nu_0})^2 - 1}{(\frac{\nu}{\nu_0})^2 + 1}$$

$$\frac{v}{c} = \frac{1.001636 - 1}{1.001636 + 1}$$

$$v = c \times \frac{1.001636 - 1}{1.001636 + 1} \approx 490km * s^{-1}$$

Answer: D

9. A stable open cluster of about $N = 1000$ sun-like stars has an angular diameter of $\theta = 30$ arc minutes and distance of $d = 500$ pc. Assuming the cluster can be approximated by a sphere of uniform density, estimate the average velocities of stars in the cluster.

   The gravitational potential energy of a sphere of uniform density and radius r is

$$U_{sphere} = -\frac{3}{5}\frac{GM_{sphere}^2}{r}$$

(a) 507 m/s
(b) 643 m/s
(c) 894 m/s
(d) 1021 m/s
(e) 771 m/s

Solution:
First, we must find the radius of the cluster. Using trigonometry, we get:

$$r = \theta \cdot d \; r = 1.35 \cdot 10^{17} \text{ m}$$

Now, we will use the radius obtained to calculate the velocity of the cluster using the Virial Theorem:

$$-2\langle K \rangle = \langle U \rangle$$

$$-2 \cdot 1000 \cdot \frac{M_\odot \cdot \langle v^2 \rangle}{2} = -\frac{3}{5}\frac{G(1000 \cdot M_\odot)^2}{r}$$

Finally, solving for $\langle v \rangle$:

$$\langle v \rangle = 771 \text{ m/s}$$

Answer: E

10. What would happen to the analemma of the Sun if the obliquity of the Earth's orbit suddenly went to zero degrees and its eccentricity remained unchanged?

(a) The anallema would be perfectly symmetric in both axes and would have the shape of an "8".
(b) The analemma would look like a dot.
(c) The analemma would be the arc of a great circle.
(d) The analemma would look like a circle.
(e) The analemma would be a spherical triangle.

Solution:
In this case, since the obliquity is equal to zero, the Sun is always at the Celestial Equator, so its declination is always equal to zero degrees. However, since there is still a non-zero eccentricity, so the right ascension still varies throughout the year for the same reference civil time. Since the Equator is a great circle, the analemma looks like the arc of a great circle.

Answer: C

11. Let $T_{\odot,C}$ and $T_{\odot,S}$ be the temperatures at the core and the surface of the sun, respectively. Similarly, let $T_{A,C}$ and $T_{A,S}$ be the temperatures at the core and surface of the red giant Arcturus, and let $T_{S,C}$ and $T_{S,S}$ be the temperatures at the core and surface of the white dwarf Sirius B. Which of the following inequalities is true?

(a) $\frac{T_{\odot,C}}{T_{\odot,S}} < \frac{T_{A,C}}{T_{A,S}} < \frac{T_{S,C}}{T_{S,S}}$

(b) $\frac{T_{\odot,C}}{T_{\odot,S}} < \frac{T_{S,C}}{T_{S,S}} < \frac{T_{A,C}}{T_{A,S}}$

(c) $\frac{T_{A,C}}{T_{A,S}} < \frac{T_{\odot,C}}{T_{\odot,S}} < \frac{T_{S,C}}{T_{S,S}}$

(d) $\frac{T_{S,C}}{T_{S,S}} < \frac{T_{\odot,C}}{T_{\odot,S}} < \frac{T_{A,C}}{T_{A,S}}$

(e) $\frac{T_{S,C}}{T_{S,S}} < \frac{T_{A,C}}{T_{A,S}} < \frac{T_{\odot,C}}{T_{\odot,S}}$

Solution:

The answer is D. Since Sirius B is a white dwarf, its interior temperature is close to uniform, so it has the smallest temperature ratio between its core and surface. Now, Arcturus is a red giant, so its surface temperature is less than the Sun's, while its core temperature will be greater. Thus, Arcturus will have the greatest temperature ratio.

Answer: D

12. The spectral line H_α in the spectrum of a star is recorded as having displacement of $\Delta\lambda = 0.043 \times 10^{-10}$ m. At rest, the spectral line has a wavelength of $\lambda_0 = 6.563 \times 10^{-7}$ m. Calculate the period of rotation for this star, if it is observed from its equatorial plane. We also know: $R_{star} = 8 \times 10^5$ km.

(a) 29.59 days

(b) 14.63 days

(c) 21.15 days

(d) 34.39 days

Solution:

Doppler for the H_α spectral line:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c}$$

Thus:

$$v = \frac{c\Delta\lambda}{\lambda_0}$$

The star is rotating:

$$v = \omega R_{star}$$

,where $\omega = \frac{2\pi}{T}$.

$$\frac{2\pi R_{star}}{T} = \frac{c\Delta\lambda}{\lambda_0}$$

So, the period of rotation for our star is:

$$T = \frac{2\pi R_{star}\lambda_0}{c\Delta\lambda} = \frac{2 \cdot \pi \cdot 8 \cdot 10^8 \cdot 6.563 \cdot 10^{-7}}{3 \cdot 10^8 \cdot 0.043 \cdot 10^{-10}} = 29.59 \quad days$$

Answer: A

13. The reflector telescope built by Sir Issac Newton was a $f/5$ telescope and had a primary mirror of diameter 30mm. He used an eyepiece with a focal length of 5mm. What is the focal length and magnification obtained by this telescope?

(a) 150mm, 30×
(b) 300mm, 15×
(c) 300mm, 30×
(d) 150mm, 15×

Solution:

$f/5$ means the focal length is 5 times the diameter i.e. 150mm. The magnification is $m \approx \frac{f_o}{f_e}$ where f_o and f_e are the objective and eyepiece focal length respectively.

Answer: A

14. Take a look at the following image:

Three Messier objects are circled in the image. Select the alternative that correctly matches each object with its type.

(a) 1 - Open cluster; 2 - Open cluster; 3 - Nebula.
(b) 1 - Open Cluster; 2 - Nebula; 3 - Galaxy.
(c) 1 - Galaxy; 2 - Nebula; 3 - Globular cluster.
(d) 1 - Open cluster; 2 - Galaxy; 3 - Globular cluster.
(e) 1 - Open cluster; 2 - Nebula; 3 - Open cluster.

Solution:

Object 1 is the open cluster Pleiades (M45), in the constellation *Taurus*.

Object 2 is the Crab Nebula (M1), in the constellation *Taurus*.

Object 3 is the Shoe-Buckle Cluster (M35), which is an open cluster in *Gemini*.

Answer: E

15. An interesting phenomena that happens in the Solar System is the capture of comets in the interstellar medium. Assume that a comet with a mass of $7.15 * 10^{16}$ kg is captured by the solar system. The perihelion of this comet's orbit after it is captured is equal to 4.64 AU, and its velocity with respect to the Sun before being captured by the Solar System was very small. Calculate the velocity of the comet at the perihelion.

(a) 87.1 km/s
(b) 45.9 km/s
(c) 5.67 km/s
(d) 105.4 km/s
(e) 19.6 km/s

Solution:

First, it is important to determine the shape of the comet's orbit.

When the comet is still far from the Sun, its velocity is very small, so its kinetic energy is negligible. At this point, the distance to the Sun is large, so the potential gravitational energy is also close to zero. Therefore, since the total mechanical energy is zero, the orbit is parabolic.

It is important to highlight that the velocity was not exactly zero and the distance to the Sun was not infinite, so the mechanical energy might not be exactly equal to zero, but it is close enough to assume that the orbit is parabolic or at least almost parabolic.

For parabolic orbits, since the total mechanical energy is equal to zero, it is possible to use the following formula to calculate the velocity of a body in a parabolic orbit:

$$v = \sqrt{\frac{2 * G * M}{r}}$$

$$v = \sqrt{\frac{2 * 6.673 * 10^{-11} * 1.989 * 10^{30}}{4.64 * 1.496 * 10^{11}}}$$

$$v = 1.96 * 10^4 \; m/s$$

$$v = 19.6 \; km/s$$

Answer: E

16. In a certain day, when it is 0h UT, the sidereal time of Prime Meridian is 5h 56min 9.4s. For this day, with start and end based on UT, find the civil time of Chicago, whose longitude and time zone are respectively, 87.65004722° W and UT-6, when the sidereal time there is 20h. The difference between solar time and sidereal time **SHOULD** be accounted for.

   (a) 14h 1min 32s
   (b) 13h 26min 17s
   (c) 14h 36min 47s
   (d) 14h 0min 43s
   (e) 13h 51min 11s

Solution:

We need to find the civil time of Chicago (UT-6) when the Sidereal Time of Chicago is 20h, for the day that begins when at 0h UT, the Sidereal Time of the Prime Meridian is 5h 56min 9.2s (5.9358889 sidereal hours).

The trick to solve this problem is to treat solar time and sidereal time with different units. Here, we will use sidereal hours and solar hours.

We know the difference between the sidereal times at any point in time:

$$\Delta = \lambda_{Chicago} - \lambda_{PM}$$

$$\Delta = 5.84333648 \; sidereal \; hours$$

Using this, we can find the sidereal time of the Prime Meridian when the Sidereal Time of Chicago is 20h.

Hence, we should add the Difference obtained to the Chicago Sidereal Time to obtain the Prime Meridian Sidereal Time when the former is 20h. For simplicity, we will label this instant as (2). We will label 0h UT as (1).

$$ST_{PM,2} = mod(20 \; sidereal \; hours + 5.84333648 \; sidereal \; hours, 24)$$

$$ST_{PM,2} = 1.84333648 \; sidereal \; hours$$

Hence, the time elapsed between (1) and (2) is:

$$\Delta T = 1.84333648 + (24 - 5.9358889)\ sidereal\ hours$$

$$\Delta T = 19.90744759\ sidereal\ hours$$

Converting the Time Elapsed into solar hours:

$$\Delta T = 19.90744759\ sidereal\ hours \cdot \frac{23.93446972\ solar\ hours}{24\ sidereal\ hours}$$

$$\Delta T = 19h\ 51min\ 11s$$

We also know that:

$$\Delta T = CT_{PM,2} - CT_{PM,1}$$

Hence,

$$CT_{PM,2} = 19h\ 51min\ 11s$$

For Chicago, we have:

$$CT_{Chicago,2} = 19h51min11s - 6h0min0s\ CT_{Chicago,2} = 13h51min11s$$

Answer: E

17. Consider the following horrifying scenario. The Sun has become a Red Giant, and its radius is doubling every 100 years. Rank the following of humanity's concerns in order of immediate importance.

I: Orbital decay due to direct contact between the Earth and Sun

II: As the distance between the Earth and Sun shrinks, the Earth will enter the Sun's Roche limit and start to be ripped apart

III: Orbital decay due to tidal effects on the Sun's outer atmosphere, the same way the Moon loses energy when forming the Earth's tides

IV: Runaway greenhouse effect due to extreme temperatures, leading to the Earth becoming a hot, Venus-like planet with no habitability

(a) III, IV, II, I
(b) IV, III, II, I
(c) IV, II, III, I
(d) IV, III, I, II

Solution:

This was a bit of a trick question focused on the Roche limit. Because the Roche limit has to do with the strength of the gravitational force exerted by the larger body, and because the Sun's mass will remain the same when it expands to become a red giant, the Roche limit does not change. Thus, the Earth will never be ripped apart due to tidal forces, and that makes (II) the least important choice.

Otherwise, the ordering of the other dangers is pretty straightforward: the Earth's temperature would immediately start to rise and the atmosphere would be ruined, followed by orbital decay from afar (tidal effects) which must logically come before direct contact of the Sun and Earth.

Therefore the answer is temperature → early orbital decay → direct contact → Roche limit (which is essentially irrelevant).

CORRECTION: We realize now that the statement "the same way the Moon loses energy when forming the Earth's tides" is not accurate, and the Earth actually is the one losing energy in this interaction, because of its rotation rate outpacing the Moon's orbital speed. This may have confused some about humanity's priority of answer III. Some may try to put it closer to the bottom of the list because with the Earth and Sun, the Sun's slower rotation speed would mean the Earth does lose energy, counter to the example given in the answer.

Since C is the only answer which puts III at a lower priority than the correct answer D, we will also accept C as an answer. (It is definitely not the correct answer, but if a student were to be confused by our mistake we don't expect them to find an optimal-yet-wrong solution).

Answer: D or C

18. TESS Object of Interest (TOI) 402.01 has an orbital period of 4.756 ± 0.000023 (days) and was last observed to transit on 2139.1 ± 0.0027008 (in TESS Julian days, i.e. BJD - 2457000). For follow-up observation, we would like to predict the next transit – this would be the 23rd transit since the last observation. In TESS Julian days, when will the next transit occur?
    (a) 2243.732 ± 0.0027238 TESS JD
    (b) 2248.488 ± 0.000529 TESS JD
    (c) 2248.488 ± 0.0027238 TESS JD
    (d) 2248.488 ± 0.0032298 TESS JD

Solution:

The predicted epoch of the next transit is $(2139.1 \pm 0.0027008) + 23 * (4.756 \pm 0.000023) = 2248.488 \pm 0.0032298$. (The error on the last observation gets added to 23 times the error on the period to get the final period.)

Answer: D

19. Jupiter has a mass of 1.90×10^{27} kg and a radius of 7.15×10^{7} m. To the closest order of magnitude, estimate the pressure at the center of Jupiter, in Megabars.

(a) 0.1
(b) 1
(c) 10
(d) 100

Solution:
From hydrostatic equilibrium:
$P_c \approx 2/3\pi G\rho^2 R^2 \approx 2/3\pi G(M/(4/3\pi R^3))^2 R^2 = 3/(8\pi)GM^2/R^4 \approx 11$ Mbars
1 bar $= 10^5$ Pa.

Answer: C

20. Which of the following statements is wrong?

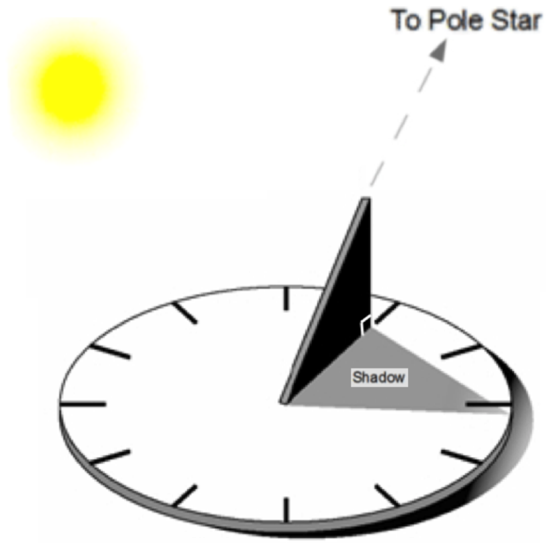
(a) It is believed that elements with atomic number greater than that of iron are formed mostly by the explosion of supernovas.
(b) What holds a star together is the hydrostatic equilibrium between pressure and gravity.
(c) The granulations of the Sun happen on its corona.
(d) Protostars are actually not stars because their main source of heat is not fusion.
(e) The earlier type the main-sequence star, the more massive it is.

Solution:
The granulations of the Sun actually happen on the photosphere of the Sun. This phenomena is the result of of the convection currents of plasma in the convection zone of the Sun. The top of each convective cell on the photosphere produces a granule, which results in the granulation of the Sun.

Answer: C

21. Consider a horizontal sundial where the triangular gnomon rises at an angle equal to the sundial site's latitude, $\phi = 38°$. If the area of the triangular gnomon is 2 m^2, what would be the area of the shadow in m^2 three hours after the noon in the first day of spring (vernal equinox)?

(a) 3.0
(b) 3.5
(c) 2.5
(d) 1.5
(e) 4.0



Solution:

The area of the shadow can be calculated by $A \cot a_\odot \sin A_\odot$ where A is the area of the triangular gnomon. Also, $a_\odot$ and $A_\odot$ are altitude and azimuth of the sun, respectively. Given that the declination of the sun is zero, so, $\sin a_\odot = \cos\phi \cos H_\odot$ and $\sin A_\odot = \sin H_\odot / \cos a_\odot$ where $H_\odot$ is the hour angle of the sun. Therefore, the area of the shadow is $A \tan H_\odot / \cos\phi = 2 \times \tan(45°)/\cos(38°)\ m^2 = 2.5\ m^2$.

Answer: C

22. Consider an eclipsing binary star system observed (in some fixed band) to have a **combined** apparent magnitude of 5.67. During the secondary transit, the second star is totally eclipsed by the first star, and the apparent magnitude dims to 6.28. What percent of the combined flux is produced by the **second** star?
    (a) 10.8%
    (b) 43.0%
    (c) 57.0%
    (d) 89.2%

Solution:

Total magnitude, $m_T = 5.67$. During the eclipse, all light comes from the first star (star A), so $m_A = 6.28$. A difference in magnitudes can be converted to a flux ratio via $m_A - m_T = -2.5\log_{10}(\frac{I_A}{I_T})$. The percent of the flux from star B is $\frac{I_B}{I_T} = 1 - \frac{I_A}{I_T}$.

Answer: B

23. An astronomer observes that a Solar type star has an apparent V magnitude of 6.73 when seen from the Earth. Assuming that the average interstellar extinction in V is 1.00 mag/kpc, determine the distance between this star and the Solar system.

(a) 11.5 pc
(b) 49.5 pc
(c) 34.2 pc
(d) 23.7 pc
(e) 18.9 pc

Solution:

It is possible to use the following formula, in which d is the distance, a_V is the interstellar extinction, m is the apparent magnitude and M is the absolute magnitude:

$$m - M = 5 * log(d) - 5 + a_V * d$$

Plugging in the values (1.00 mag/kpc is equal to 10^{-3} mag/pc):

$$6.73 - 4.83 = 5 * log(d) - 5 + 10^{-3} * d$$

$$6.9 = 5 * log(d) + 10^{-3} * d$$

This equation cannot be solved analytically, so it is necessary to use numerical methods. The simplest approach is to guess a value for the distance, plug it in, and keep increasing or decreasing the value until the right side of the equation is equal to 6.9.

Another approach is to use the scientific calculator to perform an iteration. The first step of the iteration is to rearrange the formula:

$$d = 10^{\frac{6.9 - 10^{-3} * d}{5}}$$

Next, guess a distance and press the "=" button. Then, type the formula on the right side of the equation using the "ANS" button as d. Keep pressing "=" until the result does not change anymore. This number is the distance d.

Using any of the two methods, the result is that the distance is equal to 23.7 pc.

Answer: D

24. 1. The temperature of the Sun is 5000K while that of Sirius is 10000K. Which star has a higher integrated radiance i.e. net energy emitted per unit time per unit area?

(a) Sun

(b) Sirius

(c) Depends on value of the respective radii

(d) The integrated radiance is equal.

Solution:

The integrated radiance is σT^4 from Stefan-Boltzman law.

Answer: B

25. Suppose a spaceship is attempting a slingshot maneuver on a gas giant with mass 100 times that of the spaceship. Because the spaceship somewhat entered the planet's atmosphere, kinetic energy was not conserved—only momentum. What is the ratio of the spaceship's change in velocity to the planet's change in velocity, $\frac{\Delta v_s}{\Delta v_p}$?

(a) 10

(b) 100

(c) -10

(d) -100

Solution:

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$
$m_1v_1 = m_1u_1 + m_2u_2 - m_2v_2$
$v_1 - u_1 = \frac{m_2}{m_1}(u_2 - v_2)$
$\Delta v_1 = -100\Delta v_2$

Where body 1 is the spaceship, and body 2 is the planet.

Answer: D

26. Let's imagine that our Universe would be filled with basketballs, each having a mass of $m_b = 0.62$ kg. What would be the necessary numerical density (n_b) of basketballs in the Universe such that the mass density of the basketballs would equal the current critical density of our Universe?

(a) 1.5×10^{-26} balls/m^3

(b) 1.7×10^{26} balls/m^3

(c) 1.5×10^{-27} balls/m^3

(d) 1.7×10^{27} balls/m^3

Solution:

Critical density of the Universe:

$$\rho_0 = \frac{3H_0^3}{8\pi G} = 9.31 \times 10^{-27} \quad kg/m^3$$

The basketballs are uniformly distributed in a volume V:

$$\rho = \frac{m}{V} = \frac{Nm_b}{V} = n_b m_b$$

Thus, because we assume $\rho = \rho_0$:

$$n_b = \frac{\rho}{m_b} = \frac{\rho_0}{m_b} = 1.5 \times 10^{-26} \quad balls/m^3$$

Answer: A

27. **(CANCELED)** When binary systems are really close together, they can execute an accretion process, in which one star (called the primary star) "eats" the mass of the other (called the secondary star), whose mass spirals down into the primary star, creating an accretion disk!

For an accretion disk with the outer edge $3R$ from the center of the primary star (radius R and mass M), calculate the energy lost by a test mass (mass m) where it touches the primary star from where it first enters the accretion disk.

Consider the orbits to be Keplerian.

(a) $\frac{GMm}{R}$
(b) $\frac{1}{2}\frac{GMm}{R}$
(c) $\frac{5}{2}\frac{GMm}{R}$
(d) $\frac{1}{3}\frac{GMm}{R}$
(e) $\frac{3}{4}\frac{GMm}{R}$

Solution:

Let the total energy when the test mass enters the disk be:

$$E_o = -\frac{GMm}{6R}$$

and the total energy when the test mass touches the star be:

$$E = -\frac{GMm}{2R}.$$

Hence, the difference between the two will be:

$$E_o - E = \frac{GMm}{2R} - \frac{GMm}{6R}$$
$$E_o - E = \frac{1}{3}\frac{GMm}{R}$$

Answer: D

On test day, item (d) was $\frac{2}{3}\frac{GMm}{R}$ which was incorrect.

28. An often-repeated fun fact is that humans produce more power per unit volume than stars. If the sun were the same size, but it produced the same amount of power per unit volume as a human, what would its surface temperature be? Assume the "average human" produces 100 watts of power and has a volume of 66400 cubic centimeters.

(a) 3500 K
(b) 10000 K
(c) 25000 K
(d) 40000 K
(e) 50000 K

Solution: Using the numbers from the problem, the average human produces

$$u = \frac{100}{66400 \times 10^{-6}} = 1506\,\mathrm{W/m^3}.$$

The volume of the sun is $\frac{4}{3}\pi R_\odot^3$, so its new power output would be $P = \frac{4}{3}\pi R_\odot^3 u$. To find the equilibrium temperature, we use

$$P = \sigma A T^4 = 4\pi\sigma R_\odot^2 T^4.$$

Solving for T, we get

$$T = \sqrt[4]{\frac{R_\odot u}{3\sigma}} = \sqrt[4]{\frac{(6.96 \times 10^8)\,(1506)}{3\,(5.67 \times 10^{-8})}} = 49823 \approx \boxed{50000\,\mathrm{K}}.$$

Answer: E

29. Given that the redshift of cosmic microwave background (CMB) is 1100, what was the temperature of the Universe when photon decoupled from matter and neutral hydrogen started to get formed? The present temperature of the CMB is 2.73 K.

(a) 10000 K
(b) 30000 K
(c) 3000 K
(d) 1000 K
(e) 300 K

Solution:

$2.73 \times (1 + z) \sim 3000\ K$

Answer: C

30. Where and when should we place a radio telescope such that, when combined with a radio telescope on Earth, the system could "see" the supermassive black hole in Sculptor's Galaxy (NGC 253)?

Sculptor's Galaxy's supermassive black hole's mass is estimated to be around $5 \cdot 10^6 M_\odot$, and its distance is estimated to be around 3.5 Mpc.

Out of the options below, pick the one closest to the estimate you obtain, rounding up. Consider the energy of a radio wave to be around 10^{-5} eV.

Use the following formula to estimate the angular resolution: $\theta = \frac{\lambda}{D}$

(a) On the Moon when it is at its apogee.
(b) On Mars when it is in conjunction.
(c) On Venus when it is in its greatest elongation.
(d) On one of Jupiter's moons when it is in opposition.
(e) Somewhere in the farthest points of the Oort cloud when Earth is at its perihelion.

Solution:

We know the telescope resolution formula, so let us isolate D:

$$D = \frac{\lambda}{\theta}.$$

Now, let's find θ

$$\theta = \frac{2R}{d}.$$

Where R is the radius of the supermassive black hole and d is its distance to the Sun.

Using the Schwarzchild Radius, we can find the black hole's radius:

$$R = \frac{2GM}{c^2}.$$

Now let us use the energy given to find λ:

$$E = h\nu$$

$$E = \frac{hc}{\lambda}$$

$$\lambda = \frac{hc}{E}$$

.

Now, substituting what we obtained in the resolution formula, we get:

$$D = \frac{hc^3 d}{4GME}.$$

Hence, substituting values in, we get:

$$D \approx 3 \text{ A.U..}$$

Analyzing each item:

The Moon is not even 1 A.U. from Earth, so (a) is not the answer.

Mars at its conjuction would be close at around 2.5 A.U., but it would not be enough. Hence, it cannot be (b).

Venus's semi-major axis (which we can treat as a radius for the purposes of this estimation) is of less than 1 A.U.. Hence, by the triangle inequality, at any point in its orbit, its distance to Earth is less than 2 A.U., which includes its greatest elongation. Thus, it cannot be (c).

Jupiter's semi-major axis (which we can treat as a radius for the purposes of this estimation) is of around 5.2 A.U.. At its opposition, its distance to Earth would be of around 4 A.U., which is a bit larger than our estimation (which is good! We want to be safe that we can resolve the black hole!). Of course, the telescope would not actually be on Jupiter, as it is a gas planet, but instead on one of its moons. Answer (d) seems the most likely so far.

The farthest points in the Oort cloud are at around 50000 A.U. from Earth, so it cannot be (e) when we have (d) as a choice.

Finally, we can say that a radio telescope placed on one of Jupiter's Moons would be the best choice, i.e., the answer is (d).

Answer: D