2 Short Questions

1. (**5 points**) Jupiter emits more energy to space than it receives from the Sun. The internal heat flux of Jupiter can be quantified by the "intrinsic" temperature of the planet T_{int}. The effective temperature T_{eff} of a planet is related to its intrinsic temperature and equilibrium temperature T_{eq} by $T_{\text{eff}}^4 = T_{\text{eq}}^4 + T_{\text{int}}^4$. Given that Jupiter's albedo is 0.5, its emissivity is 1, its average separation from the Sun is 5.2 AU, and its effective temperature is 134 K, estimate its intrinsic temperature in Kelvin. You may use the Sun's surface temperature equal to 5777 K.

Solution: Solve for equilibrium temperature, by equating solar flux falling on Jupiter to a emission of a black-body at temperature T_{eq}

$$T_{\text{eq}} = \left(\frac{R_{\text{Sun}}}{a}\right)^{1/2} \left[\frac{(1-A)}{4\epsilon}\right]^{1/4} T_{\text{Sun}}, \tag{1}$$

Here, A is the albedo, ϵ is the emissivity, a is Jupiter-Sun separation.

$$T_{\text{eq}} = \left(\frac{6.96 \times 10^8 \text{ m}}{5.2 \times 1.5 \times 10^{11} \text{ m}}\right)^{1/2} \left[\frac{(1-0.5)}{4}\right]^{1/4} 5777 \text{ K} = 102 \text{ K}. \tag{2}$$

Plug in to solve for intrinsic temperature

$$T_{\text{int}} = \left(T_{\text{eff}}^4 - T_{\text{eq}}^4\right)^{1/4} = \left[(134 \text{ K})^4 - (102 \text{ K})^4\right]^{1/4} = 121 \text{ K}. \tag{3}$$

Note: Accept slightly higher values if the student uses the Solar constant rather than Stefan-Boltzmann law to estimate the Sun's emitted flux.

2. (**5 points**) The convection zone of the sun is the major region of the solar interior that is closest to the surface. It is characterized by convection currents that quickly carry heat to the surface. As a pocket of gas rises, it expands and becomes less and less dense. For it to continue to rise, the temperature gradient in the sun must be steeper than the adiabatic gradient, which is the temperature that the gas would have if it were allowed to expand without any heat input.

In the sun, the adiabatic gradient satisfies $T \propto p^{0.4}$, where T is the temperature and p is the pressure at any given point.

The bottom of the convection zone is about 200,000 kilometers beneath the surface of the sun, and has a temperature of about 2×10^6 K and a density of about 200 kg/m^3. Estimate an upper bound for the temperature of the convection zone where the density is 1.2 kg/m^3 (the density of air). You may assume the ideal gas law holds in the convective zone.

Solution: Since the temperature gradient is steeper than the adiabatic gradient, we can get an upper bound for the temperature by assuming the temperature gradient follows the adiabatic gradient exactly.

For a fixed amount of gas molecules, we have that $p \propto \frac{T}{V}$ by the ideal gas law. Also, the adiabatic gradient has $T \propto p^{0.4}$, so we get

$$T \propto \left(\frac{T}{V}\right)^{0.4} = \frac{T^{0.4}}{V^{0.4}}.$$

Using the fact that V is inversely proportional to the density ρ, we have $T \propto T^{0.4}\rho^{0.4}$. Dividing both sides by $T^{0.4}$, we get $T^{0.6} \propto \rho^{0.4}$, so

$$T \propto \rho^{2/3}.$$

Plugging in the numbers we are given, we have that the temperature when the density is equal to that of air is

$$(2 \times 10^6\,\mathrm{K}) \left(\frac{1.2\,\mathrm{kg/m^3}}{200\,\mathrm{kg/m^3}} \right)^{2/3} = \boxed{66000\,\mathrm{K}}.$$

3. (**5 points**) Galaxies are very hard to spot, even those that are nearest to us. For instance, Andromeda, despite having an apparent magnitude of 3.44, appears very "dim" in the sky. This is because its light is very spread out, since its solid angle in the sky is so large (around 3 times that of the Sun!).

   Hence, it is often useful to use the surface magnitude of a galaxy, defined as the magnitude that a certain solid angle of that galaxy has. It is usually measured in $mag/arcmin^2$.

   Show that, in a non expanding universe, the surface magnitude is independent of the distance to the galaxy.

Solution: Let us begin with

$$m_{unit} - m_{total} = -2.5 \log \left(\frac{F_{unit}}{F_{total}} \right)$$

$$m_{unit} - m_{total} = -2.5 \log \left(\frac{\Omega_{unit}}{\Omega_{total}} \right)$$

$$m_{unit} = -2.5 \log(\Omega_{unit}) + 2.5 \log(\Omega_{total}) + m_{total}$$

Where m_{unit} is the surface magnitude, i.e, the magnitude of 1 unit of solid angle; m_{total} is the actual magnitude of the galaxy; F_{unit} is the flux from a unit solid angle; F_{total} is the total flux from the galaxy; Ω_{unit} is the unit solid angle; and Ω_{total} is the total solid angle of the galaxy.

We must show that m_{unit} does not depend on the distance d.

For this, we must proceed as follows:

$$2.5 \log(\Omega_{total}) = 2.5 \log \left(\frac{A}{d^2} \right)$$

$$2.5 \log(\Omega_{total}) = 2.5 \log(A) - 5 \log(d)$$

where A is the physical area of the galaxy and d is the distance to it.

And also, with the distance modulus equation, we can get:

$$m_{total} = M_{total} + 5 \log(d) - 5$$

where M_{total} is the absolute magnitude of the galaxy.

Putting everything together, we have:

$$m_{unit} = -2.5 \log(\Omega_{unit}) + 2.5 \log(A) + M_{total} - 5$$

which is independent of distance, as desired.

4. (**5 points**) An Earth satellite has the following position ($\vec{r}$) and velocity ($\vec{v}$) vectors at a given instant:

$$\vec{r} = 7000\hat{i} + 9000\hat{j}(\text{km})$$

$$\vec{v} = -2\hat{i} + 5\hat{j} \ (\text{km/s})$$

Calculate the eccentricity of the satellite orbit. Hint: The eccentricity of the orbit is related to total energy E and angular momentum L as $e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}$; where M is Earth's mass and m is the mass of the satellite.

Solution: Method 1: For a particle experiencing a force in the form of $F(\vec{r}) = -\frac{k}{r^2}\hat{r}$, the Laplace–Runge–Lenz vector is defined as follows:

$$\vec{A} = \vec{P} \times \vec{L} - mk\hat{r}$$

where $\vec{P}$ is the linear momentum and $\vec{L}$ is the angular momentum. The eccentricity can be calculated from the Laplace–Runge–Lenz vector:

$$e = \frac{|A|}{mk}$$

$$e = \frac{|m\vec{v} \times (\vec{r} \times m\vec{v}) - mGMm\hat{r}|}{mGMm}$$

$$e = |\frac{\vec{v} \times (\vec{r} \times \vec{v})}{GM} - \hat{r}|$$

$$e \sim 0.53$$

Method 2: One can also compute the energy ($\frac{E}{m} = \frac{v^2}{2} - \frac{GM}{r}$) and the angular momentum ($\frac{L}{m} = |\vec{r} \times \vec{v}|$) of the orbit and use the following equation to get the eccentricity:

$$e = \sqrt{1 + 2\frac{\frac{E}{m}(\frac{L}{m})^2}{G^2M^2}} \sim 0.53$$

5. (**5 points**) An astronomer who lives in Chicago ($\phi = 41.88^o N$; $\lambda = 87.63^o W$) was very bored during the day of the winter solstice in the Northern hemisphere, so he started thinking about the sunset. The astronomer could not wait to see the sunset on that day. Considering that the true solar time at his location was 2:30 pm, how long did he have to wait to see the sunset? The declination of the sun on winter solstice is $\delta = -23.44^o$.

Solution:

Consider the spherical triangle formed by the Sun, the zenith and the north pole. Let H be the hour-angle, h be the Sun's altitude and δ be the declination of the Sun.

$$\cos(H) = \frac{\sin(h) - \sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)}$$

Since $h = 0^o$ for the sunrise and the sunset:

$$\cos(H) = -\tan(\phi)\tan(\delta)$$

For the winter solstice in the Northern hemisphere, $\delta = -23.44^o$. We get $H = \pm 67.11^o = 4.474\,h$.

Therefore, the Sun rises 4.474 hours before true solar noon and sets 4.474 hours after true solar noon. Let δt_1 be the time the astronomer has to wait for sunset after $2:30\,\text{pm}$.

$$\Delta t_1 = 12:00pm + 4.474h - 2:30pm$$
$$\Delta t_1 = 1h\,58min$$

$\therefore$ The astronomer would have to wait 1h 58min to see the sunset.