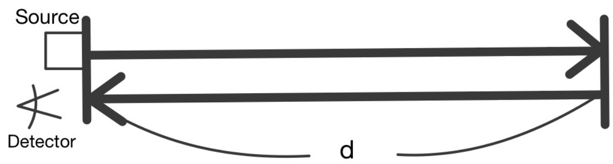
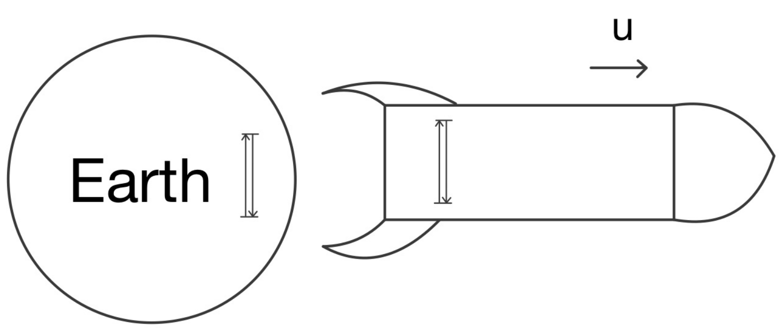
3 Medium Questions

1. (**20 points**) To measure the time accurately outside the Earth, the engineers build a special clock, with the design as follows: there is a source of light that sends light particles (photons) straight to the reflector that is located at the distance d away from the source. The reflector sends the photons back to their starting point, where there is a detector. This can measure the time accurately, because the speed of light c is constant everywhere. Then a group of engineers built a spaceship with this special clock inside. This spaceship with a clock started to move really fast at the speed u. While the observer in the spaceship reported no issues with the clock inside the spaceship, the observer on the Earth has noticed that the clock is functioning differently in a fast moving spaceship than it is on Earth.





(a) Given that the clock is at rest, what is the total traveling time (Δt_E) of a photon from its source back to the detector?

(b) What is the total distance traveled by a photon d_γ from the source back to the detector on the spaceship moving at the speed u away from the Earth? (Here, we denote that the total traveling time of the photon as Δt_S)

(c) What is the total time Δt_S of a photon as it travels from the source to the detector on the moving spaceship? Answer in terms of d, c, and $\beta = \frac{v}{c}$.

(d) If we relate Δt_E (non-moving frame) to Δt_S (moving frame), as follows: $\Delta t_S = \gamma \Delta t_E$, what does γ equal to? What is significant about the range of γ?

(e) So far, we have only analyzed the motion on the perspective of an observer on the Earth. From the perspective of an observer on the moving spaceship, how do the time on the spaceship $\Delta t_S{}'$ and the time on the Earth $\Delta t_E{}'$ relate to each other?

(f) What can we conclude about the relative passing on time on two different frames that are relatively in motion to one another?

Solution:

(a) $\Delta t_E = \frac{2d}{c}$.

(b) While the photon travels from the source to the reflector, the spaceship moves by $u \cdot \frac{\Delta t_S}{2}$. The distance from the source to the reflector can be then obtained by Pythagorean theorem.

$$\frac{d_S}{2} = \sqrt{d^2 + (u \cdot \frac{\Delta t_S}{2})^2}$$

. Therefore,

$$d_S = 2\sqrt{d^2 + (u \cdot \frac{\Delta t_S}{2})^2}.$$

(c) The speed of light c is constant everywhere. Therefore, the distance traveled by light during Δt_S is $c \cdot \Delta t_S$.

From the last section,

$$2\sqrt{d^2 + (u \cdot \frac{\Delta t_S}{2})^2} = c \cdot \Delta t_S.$$

$$4d^2 + u^2 {\Delta t_S}^2 = {\Delta t_S}^2 \cdot c^2.$$

$${\Delta t_S}^2 (c^2 - u^2) = 4d^2.$$

$$\Delta t_S = \frac{2d}{\sqrt{c^2 - u^2}} = \frac{2d}{c\sqrt{1 - \frac{v^2}{c^2}}},$$

which equals to

$$\frac{2d}{c} \cdot \frac{1}{\sqrt{1 - (\frac{v}{c})^2}},$$

or

$$\frac{2d}{c} \cdot \frac{1}{\sqrt{1 - \beta^2}}.$$

(d)

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

Since no other speed can exceed the speed of light c, β is always less than 1. Therefore γ is always greater than 1.

(e)

$$\gamma {\Delta t_S}' = {\Delta t_E}'.$$

Time moves slower on the Earth, because by relativity, the "moving spaceship" observer thinks that the Earth is flying away with a speed u.

(f) Time moves slower in a moving frame when observed from a frame at rest.

2. (**30 points**) You want to send a rocket with an instrument to analyze the atmosphere of Jupiter. In order to get there, you decide to use a Hohmann transfer orbit. $r_E = 1$ AU and $r_J = 5$ AU represent the radii of Earth's and Jupiter's circular orbits around the Sun, respectively. m, M_E, M_J, and M_S represent the masses of your rocket, Earth, Jupiter, and Sun, respectively. Ignore planetary gravitational influences. You may use any other variables you would like if you clearly define them first. Refer to the figures at the end of the question. Show your work for all derivations.

(a) Explain which two (relevant) physical quantities are conserved during this transfer orbit. Write down their statements mathematically.

(b) How long will it take to reach Jupiter?

(c) Halfway through its path to Jupiter, an unrealistic comet passes right next to your rocket and its icy tail freezes your rocket fuel. What is the **maximum** amount of time that you can afford to pass until you need the fuel to be once again unfrozen?

(d) Knowing that this comet will come in the way, your colleague suggests a bi-elliptic transfer orbit instead, with a peak distance of $12r_E$. Write equations describing how long it will now take to reach Jupiter. Will this solution always avoid the comet?

Now that you've compared the orbital times, you want to try and calculate the difference in efficiency.

(e) Derive the δv for each orbital transition in the Hohmann transfer, and sum them to find the total δv.

(f) Derive the δv for each orbital transition in the Bi-elliptic transfer, and sum them to find the total δv.

(g) Factoring in all your previous results, which transfer would you like to use? Why?

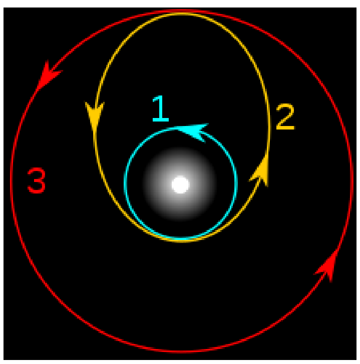


Figure 1: Hohmann Transfer

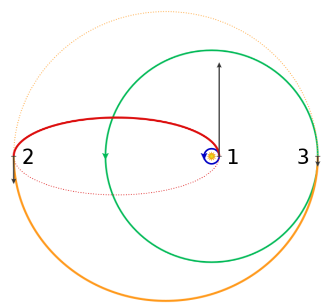


Figure 2: Bi-elliptic Transfer

Solution:

(a) The relevant preserved quantities are energy and angular momentum.

Preservation of angular momentum is given by the following: $L = mv_E r_E = mv_J r_J$ where v_E represents the rocket velocity at the Earth and v_J at Jupiter. More generally, $L = mv_1 r_1 = mv_2 r_2$.

Preservation of energy is given by the following: $E_{TOT} = \frac{mv_E^2}{2} - \frac{GmM_S}{r_E} = \frac{mv_J^2}{2} - \frac{GmM_S}{r_J}$. More generally, $E_{TOT} = \frac{mv_1^2}{2} - \frac{GmM_S}{r_1} = \frac{mv_2^2}{2} - \frac{GmM_S}{r_2}$.

(b) This can be calculated by first specifying the parameters of the rocket's elliptical orbit, and then using Kepler's third law.

Key: use Kepler's third law. The semi-major axis is the average of the minimum and maximum distances from the sun. $a = \frac{r_E + r_J}{2}$. If this is specified in AU, we can use $P^2 = a^3$ where P is in years. Thus, $P = \left(\frac{r_E+r_J}{2}\right)^{\frac{3}{2}}$. However, we only want to take half of this orbital period, since we are stopping at Jupiter rather than coming all the way back to Earth's orbital radius. Thus, we simply take half of this value:

$$T = \frac{1}{2}\left(\frac{r_E + r_J}{2}\right)^{\frac{3}{2}}$$

.

(c) You can take an unlimited amount of time to wait for the fuel to unfreeze. This is because we are ignoring gravitational effects from planets, and thus the rocket will constantly stay in the elliptical orbit with Jupiter's orbit at apoapsis and Earth's orbit at periapsis. Whenever the fuel unfreezes, we would wait from that point on for the rocket to reach apoapsis before firing engines for δv_2.

(d) Remember that a bi-elliptic transfer has one free parameter: the joint apoapsis X AU from the sun, or the max distance from the sun the rocket will reach. Thus, we immediately know this solution will not always avoid the comet. If we set the joint apoapsis to be just barely greater than Jupiter's orbit, the first elliptical orbit will be almost identical to the Hohmann transfer orbit.

Using similar logic from part b, the time to reach Jupiter would be $T = \frac{1}{2}a_1^{\frac{3}{2}} + \frac{1}{2}a_2^{\frac{3}{2}}$.

$$a_1 = \frac{12r_E + r_E}{2} = 6.5r_E$$
$$a_2 = \frac{12r_E + r_J}{2} = 6r_E + \frac{r_J}{2}$$

(e) Key: use the vis-viva equation.

Vis-viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Circular orbit 1 radius: r_E

Semi-major axis of elliptical orbit:

$$a = \frac{r_E + r_J}{2}$$

Circular orbit 2 radius: r_J

Thus, Velocity at Earth in circular orbit =

$$\sqrt{\frac{GM}{r_E}}$$

Velocity at Earth in elliptical orbit =

$$\sqrt{GM\left(\frac{2}{r_E} - \frac{2}{r_E + r_J}\right)}$$

Velocity at Jupiter in elliptical orbit =

$$\sqrt{GM\left(\frac{2}{r_J} - \frac{2}{r_E + r_J}\right)}$$

Velocity at Jupiter in circular orbit =

$$\sqrt{\frac{GM}{r_J}}$$

Now, to find the δv at each stage, compute the differences in velocities between each interface between the elliptical and circular orbits.

$$\delta v_1 = \sqrt{GM\left(\frac{2}{r_E} - \frac{2}{r_E + r_J}\right)} - \sqrt{\frac{GM}{r_E}}$$

$$\delta v_2 = \sqrt{\frac{GM}{r_J}} - \sqrt{GM\left(\frac{2}{r_J} - \frac{2}{r_E + r_J}\right)}$$

$$\delta v_{tot} = \sqrt{GM}\left(\sqrt{\frac{2}{r_E} - \frac{2}{r_E + r_J}} - \sqrt{\frac{1}{r_E}} + \sqrt{\frac{1}{r_J}} - \sqrt{\frac{2}{r_J} - \frac{2}{r_E + r_J}}\right)$$

(f) Key: use the vis-viva equation.

Vis-viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Semi-major axis of ellipse 1:

$$a_1 = \frac{r_E + X}{2}$$

Semi-major axis of ellipse 2:

$$a_2 = \frac{r_J + X}{2}$$

Velocity at Earth in Earth's orbit:

$$\sqrt{GM}\sqrt{\frac{1}{r_E}}$$

Velocity at Earth in elliptical orbit 1:

$$\sqrt{GM}\sqrt{\frac{2}{r_E} - \frac{1}{a_1}}$$

Velocity at joint apoapsis in elliptical orbit 1:

$$\sqrt{GM}\sqrt{\frac{2}{X} - \frac{1}{a_1}}$$

Velocity at joint apoapsis in elliptical orbit 2:

$$\sqrt{GM}\sqrt{\frac{2}{X} - \frac{1}{a_2}}$$

Velocity at Jupiter in elliptical orbit 2:

$$\sqrt{GM}\sqrt{\frac{2}{r_J} - \frac{1}{a_2}}$$

Velocity at Jupiter in Jupiter's orbit:

$$\sqrt{GM}\sqrt{\frac{1}{r_J}}$$

Using the above equations, we can write all the δv's.

Speed boost to enter elliptical orbit 1:

$$\delta v_1 = \sqrt{GM}\left(\sqrt{\frac{2}{r_E} - \frac{1}{a_1}} - \sqrt{\frac{1}{r_E}}\right)$$

Speed boost to enter elliptical orbit 2:

$$\delta v_2 = \sqrt{GM}\left(\sqrt{\frac{2}{X}-\frac{1}{a_2}}-\sqrt{\frac{2}{X}-\frac{1}{a_1}}\right)$$

Speed reduction to enter Jupiter's orbit:

$$\delta v_3 = \sqrt{GM}\left(\sqrt{\frac{2}{r_J}-\frac{1}{a_2}}-\sqrt{\frac{1}{r_J}}\right)$$

Total δv, replacing $X = 12r_E$:

$$\delta v_{tot} = \sqrt{GM}\left(\sqrt{\frac{2}{r_E}-\frac{1}{a_1}}-\sqrt{\frac{1}{r_E}}+\sqrt{\frac{1}{6r_E}-\frac{1}{a_2}}-\sqrt{\frac{1}{6r_E}-\frac{1}{a_1}}+\sqrt{\frac{2}{r_J}-\frac{1}{a_2}}-\sqrt{\frac{1}{r_J}}\right)$$

(g) This could be algebraically solved. Much more simply, plug in $r_E = 1$ AU and $r_J = 5$ AU to the time and velocity equations. The Hohmann transfer is both faster and more efficient in terms of energy use (δv).

3. (**30 points**) A space station of mass m is orbiting a planet of mass M_0 on a circular orbit of radius r. At a certain moment, a satellite of mass m is launched from the space station with a relative velocity $\overrightarrow{w}$ oriented towards the center of the planet. Assume that $w < \sqrt{\frac{GM_0}{r}}$.

   (a) Justify the shape of the satellite's orbit after launching and, for the satellite-planet system, determine the following quantities:
      (1) Satellite's velocity relative to the planet, immediately after launch, v
      (2) Total angular momentum of the satellite-planet system, $L_{P,Sat}$
      (3) Satellite's orbit semi-major and semi-minor axes, a_{Sat} and b_{Sat}
      (4) Satellite's orbit eccentricity, ϵ_{Sat}
      (5) Apogee and perigee distances, $r_{max,Sat}$ and $r_{min,Sat}$
      (6) Satellite's minimum velocity, $v_{min,Sat}$ and maximum velocity $v_{max,Sat}$ on it's orbit
      (7) Total energy of the satellite-planet system, $E_{Sat,P}$.

   (b) Determine the shape of the space station's orbit relative to the planet, after the satellite was launched.

Solution:

(a) The absolute velocity of the satellite (relatively to the planet) at the moment of launching is:

$$\overrightarrow{v} = \overrightarrow{w} + \overrightarrow{u}$$

,where u is the orbital station's velocity relatively to the planet.

$$v = \sqrt{w^2 + u^2}$$

$$u = \omega_0 r$$

,where ω_0 is the angular velocity of the space station orbiting around the planet, before the satellite was launched.

$$T_0 = \frac{2\pi r}{u} = \frac{2\pi}{\omega_0} = \frac{u}{r}$$

,where T_0 is the orbital period of the space station, before the satellite was launched.

$$\frac{mu^2}{r} = \frac{GmM_0}{r^2}$$

$$u = \sqrt{\frac{GM_0}{r}} \quad \omega_0 = \sqrt{\frac{GM_0}{r^3}}$$

Thus, we get:

$$v = \sqrt{w^2 + \frac{GM_0}{r}} > w$$

So, after the satellite was launched at point Q, it will move on an elliptical orbit, having the planet as a one of the focal points. This happens because the total energy of the satellite - planet system, calculated for the moment when the satellite is injected on its orbit is:

$$E_{Q,S-P} = \frac{mv^2}{2} - \frac{GmM_0}{r} = \frac{m}{2}(w^2 + u^2) - \frac{GmM_0}{r}$$

$$E_{Q,S-P} = \frac{m}{2}(w^2 + \frac{GM_0}{r}) - \frac{GmM_0}{r}$$

$$E_{Q,S-P} = \frac{m}{2}(w^2 - \frac{GM_0}{r})$$

$$w < \sqrt{\frac{GM_0}{r}}; \quad w^2 < \frac{GM_0}{r}; \quad E_{Q,S-P} < 0$$

Having this, we can determine the angular momentum of the space station relatively to the fixed planet, corresponding to point Q (point of injection):

$$\overrightarrow{L_{Q,Sat}} = \overrightarrow{r} \times mv = \overrightarrow{r} \times m(\overrightarrow{w} + \overrightarrow{u})$$

But the angle between $\overrightarrow{r}$ and $\overrightarrow{w}$ is 180°. Thus, $\overrightarrow{r} \times \overrightarrow{w} = 0$. So:

$$\overrightarrow{L_{Q,Sat}} = \overrightarrow{r} \times m\overrightarrow{u}$$

Moreover, the angle between $\overrightarrow{r}$ and $\overrightarrow{u}$ is 90°. Thus, we get:

$$L_{Q,Sat} = mru \quad u = \omega_0 r$$

$$L_{Q,Sat} = mr^2\omega_0 = mr^2\sqrt{\frac{GM_0}{r^3}} = m\sqrt{GrM_0}$$

Evolving on an elliptical orbit around the planet, when the satellite will reach the minimum distance from the planet, its angular momentum will be:

$$\overrightarrow{L_{P,Sat}} = \overrightarrow{r_{min,Sat}} \times m\overrightarrow{v_{max,Sat}}$$

Because the angle between $\overrightarrow{r_{min,Sat}}$ and $\overrightarrow{v_{max,Sat}}$ is 90°, we get:

$$L_{P,Sat} = mr_{min,Sat}v_{max,Sat}$$

Because the total angular momentum is conserved:

$$L_{P,Sat} = L_{Q,Sat} = L_{A,Sat}$$

$$mr_{min,Sat}v_{max,Sat} = m\sqrt{GrM_0} = mr_{max,Sat}v_{min,Sat}$$

Thus, we get:

$$r_{min,Sat}v_{max,Sat} = \sqrt{GrM_0} = r_{max,Sat}v_{min,Sat}$$

$$v_{max,Sat} = \frac{\sqrt{GrM_0}}{r_{min,Sat}}; \quad v^2_{max,Sat} = \frac{GrM_0}{r^2_{min,Sat}}$$

$$v_{min,Sat} = \frac{\sqrt{GrM_0}}{r_{max,Sat}}; \quad v^2_{min,Sat} = \frac{GrM_0}{r^2_{max,Sat}}$$

Also, the total mechanical energy of the satellite - planet system is conserved, thus:

$$E_{Q,S-P} = E_{P,S-P}$$

$$\frac{1}{2}m(w^2 + \frac{GM_0}{r}) - \frac{GmM_0}{r} = \frac{mv^2_{max,Sat}}{2} - \frac{GmM_0}{r_{min,Sat}}$$

$$(w^2 + \frac{GM_0}{r}) - \frac{2GM_0}{r} = v^2_{max,Sat} - \frac{2GM_0}{r_{min,Sat}}$$

$$w^2 - \frac{GM_0}{r} = v^2_{max,Sat} - \frac{2GM_0}{r_{min,Sat}}$$

But:

$$v^2_{max,Sat} = \frac{GrM_0}{r^2_{min,Sat}}$$

Thus:

$$w^2 - \frac{GM_0}{r} = \frac{GrM_0}{r^2_{min,Sat}} - \frac{2GM_0}{r_{min,Sat}}$$

$$(w^2 - \frac{GM_0}{r})r^2_{min,Sat} + 2GM_0r_{min,Sat} - GrM_0 = 0$$

By solving this, we get:

$$r_{min,Sat} = \frac{r(-GM_0 \pm w\sqrt{GM_0r})}{w^2r - GM_0}$$

$$r_{min,Sat} = \frac{r(GM_0 \mp w\sqrt{GM_0r})}{GM_0 - w^2r}$$

Because we are searching for the minimum value, we will assign $r_{min,Sat}$ the negative solution. Thus, the positive solution will be the value for $r_{max,Sat}$. Thus:

$$r_{min,Sat} = \frac{r(GM_0 - w\sqrt{GM_0r})}{GM_0 - w^2r}$$

$$v_{max,Sat} = \frac{r}{r_{min,Sat}}\sqrt{\frac{GM_0}{r}}$$

Thus, we get;

$$v_{max,Sat} = \frac{GM_0 - w^2r}{GM_0 - w\sqrt{GrM_0}}\sqrt{\frac{GM_0}{r}}$$

Consequently, for $r_{max,Sat}$:

$$r_{max,Sat} = \frac{r(GM_0 + w\sqrt{GM_0r})}{GM_0 - w^2r}$$

$$v_{min,Sat} = \frac{r}{r_{max,Sat}}\sqrt{\frac{GM_0}{r}}$$

Thus:

$$v_{min,Sat} = \frac{GM_0 - w^2 r}{GM_0 + w\sqrt{GrM_0}}\sqrt{\frac{GM_0}{r}}$$

From the properties of the ellipse:

$$r_{min,Sat} + r_{max,Sat} = 2a_{Sat}$$

,where a is the semi-major axis of the ellipse. Thus:

$$\frac{r(GM_0 + w\sqrt{GM_0 r})}{GM_0 - w^2 r} + \frac{r(GM_0 - w\sqrt{GM_0 r})}{GM_0 - w^2 r} = 2a_{Sat}$$

$$a_{Sat} = \frac{GrM_0}{GM_0 - rw^2} > r$$

Using the conservation laws for the total energy and total angular momentum, we prove that:

$$L_{Sat} = mb_{Sat}\sqrt{\frac{GM_0}{a_{Sat}}} = L_{P,Sat} = mv_{max,Sat}r_{min,Sat}$$

$$b_{Sat}\sqrt{\frac{GM_0}{a_{Sat}}} = v_{max,Sat}r_{min,Sat}$$

$$b_{Sat} = v_{max,Sat}r_{min,Sat}\sqrt{\frac{a_{Sat}}{GM_0}} = \sqrt{GrM_0}\cdot\sqrt{\frac{a_{Sat}}{GM_0}}$$

Thus, the semi-minor axis is:

$$b_{Sat} = r\cdot\sqrt{\frac{GM_0}{GM_0 - rw^2}}$$

Now, we can determine the eccentricity of the satellite's orbit:

$$e_{Sat} = \sqrt{1 - \frac{b_{Sat}^2}{a_{Sat}^2}} = \sqrt{1 - \frac{r^2\cdot\frac{GM_0}{GM_0 - rw^2}}{r^2\cdot\left(\frac{GM_0}{GM_0 - rw^2}\right)^2}} = \sqrt{1 - \frac{GM_0 - rw^2}{GM_0}}$$

Thus, we get:

$$e_{Sat} = w\sqrt{\frac{r}{GM_0}}$$

The total energy of the satellite-Earth system is:

$$E_{Sat-E} = -\frac{GmM_0}{2a_{Sat}}$$

,where $a_{Sat} = r\frac{GM_0}{GM_0 - rw^2}$.

$$E_{Sat-E} = -\frac{GmM_0}{2r\frac{GM_0}{GM_0 - rw^2}} = -GmM_0\frac{GM_0 - rw^2}{2rGM_0}$$

$$E_{Sat-E} = -\frac{m(GM_0 - rw^2)}{2r}$$

(b) After the satellite was launched with a radial relative velocity of $\overrightarrow{w}$ towards the planet, the space station also gained a radial velocity $\overrightarrow{W}$ oriented in the opposite direction of $\overrightarrow{w}$ because the total momentum is conserved:

$$\overrightarrow{V} = \overrightarrow{W} + \overrightarrow{u}$$
$$V = \sqrt{W^2 + u^2}$$

Conservation of momentum:

$$M\overrightarrow{W} + m\overrightarrow{w} = 0; \quad W = \frac{m}{M}w; \quad m < M; \quad W < w;$$

$$V = \sqrt{\frac{m^2}{M^2}w^2 + u^2}; \quad V < v;$$

In order to determine the shape of the orbit we need to calculate the total energy of the station-planet system:

$$E_{Sta,P} = \frac{MV^2}{2} - \frac{GMM_0}{r} = \frac{M}{2}(\frac{m^2}{M^2}w^2 + u^2) - \frac{GMM_0}{r}$$
$$E_{Sta,P} = \frac{M}{2}(\frac{m^2}{M^2}w^2 + \frac{GM_0}{r}) - \frac{GMM_0}{r}$$
$$E_{Sta,P} = \frac{1}{2}M\frac{m^2}{M^2}w^2 - \frac{1}{2}\frac{GMM_0}{r}$$
$$E_{Sta,P} = \frac{1}{2}M\frac{m^2}{M^2}\frac{GM_0}{r} - \frac{1}{2}\frac{GMM_0}{r}$$
$$E_{Sta,P} = \frac{1}{2}\frac{GMM_0}{r}(\frac{m^2}{M^2} - 1) < 0$$

Thus, after the satellite is launched, the space station will move on an elliptical orbit around the planet.