# 4 Long Questions

1. (**40 points**) In the very early universe, everything is in thermodynamic equilibrium and particles are freely created, destroyed, and converted between each other due to the high temperature. In one such process, the reaction converting between neutrons and protons happens at a very high rate. In thermal equilibrium, the relative number density of particle species is given approximately by the Boltzmann factor:

$$n_i \propto \exp\left[-\frac{E_i}{k_B T}\right],$$

where $E_i = m_i c^2$ is the rest energy. Additionally, the temperature during the radiation-dominated early universe is given by $T(t) \approx 10^{10}\,\mathrm{K}\left(\frac{t}{1\,\mathrm{s}}\right)^{-1/2}$, where $t$ is the time since the Big Bang.

   (a) (4 points) At a temperature where $k_B T \approx 0.8\,\mathrm{MeV}$, known as the *freeze-out* temperature, the neutrino interactions essentially stop, preventing further conversion between protons and neutrons.

      i. (2 points) About how long after the Big Bang did this occur?

      ii. (2 points) At the freeze-out temperature, what was the equilibrium ratio of the number density of neutrons to that of protons?

   (b) (3 points) Free neutrons are unstable, and decay into protons with a characteristic decay time of $\tau = 886\,\mathrm{s}$ (the time for which the number of neutrons drops to $1/e$ of the original amount). Given that helium nuclei only formed $t_{nuc} = 200\,\mathrm{s}$ after freezing out, what was the ratio of the number density of neutrons to that of protons when the helium nuclei formed?

   (c) (7 points) While trace amounts of several small nuclei were formed during Big Bang Nucleosynthesis (BBN), assume that all neutrons go into forming helium-4.

      i. (5 points) After the helium nuclei formed, what was the ratio of the number of helium-4 nuclei to the number of hydrogen nuclei?

      ii. (2 points) Approximating the mass of helium-4 as 4 times that of H (for this part only), what fraction of baryonic mass in the universe is helium?

*If you weren't able to solve part (c), assume reasonable values for the initial mass fractions of hydrogen and helium for future parts.*

   (d) (2 points) Albert the Astronomer claims that in older galaxies, the mass fraction of hydrogen should gradually be increasing, as neutrons slowly continue to decay into protons. Is his claim correct? If not, explain.

   (e) (7 points) Suppose a certain region of a galaxy has a density of $10^{-19}\,\mathrm{kg/m^3}$ and is composed of 70% hydrogen and 30% helium-4 by mass (ignore any heavier elements). Because the region is gravitationally bound, this density doesn't change significantly with the expansion of the universe; approximate it as constant. Assume hydrogen is converted into helium by the fusion reaction:

$$4\,{}^1\mathrm{H}^+ + 2\mathrm{e}^- \rightarrow {}^4\mathrm{He}^{2+} + 2\nu_e,$$

   where the electron $\mathrm{e}^-$ and electron neutrino $\nu_e$ are of negligible mass. ${}^4\mathrm{He}$ has a mass of $m_{He} = 3728.4\,\mathrm{MeV/c^2}$

      i. (4 points) Over the entire time since BBN, how much energy does this process release per cubic kiloparsec? Give your answer in joules per cubic kiloparsec.

      ii. (3 points) Assuming the age of the universe is 13.8 billion years, calculate the average luminosity density in solar luminosities per cubic kiloparsec.

Let's go back and explore how we arrived at the number $t_{nuc} \approx 200\,\mathrm{s}$, the time at which Big Bang nucleosynthesis began. Let's define $t_{nuc}$ as the time at which half the neutrons fused with protons into deuterium ($^2$H), as deuterium fusion is the first step in BBN. From the Maxwell-Boltzmann equation, the relative abundances of deuterium, protons and neutrons is given by

$$\frac{n_D}{n_p n_n} = 6\left(\frac{m_n k_B T}{\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{B_D}{k_B T}\right),$$

where $B_D = (m_p + m_n - m_D)\, c^2 = 2.22\,\text{MeV}$ is the energy released in a deuterium fusion reaction.

(f) (3 points) The number density of photons is given by $n_\gamma = 0.243\left(\frac{k_B T}{\hbar c}\right)^3$. Find an expression for the number density of protons $n_p$ in terms of the temperature $T$ and the baryon to photon ratio $\eta$. You may use your answer to part (b).

(g) (3 points) Find the present-day baryon to photon ratio. The CMB temperature is 2.725 K, and the present-day density parameter for baryonic matter is $\Omega_{b,0} = \frac{\rho_{b,0}}{\rho_{c,0}} = 0.04$. $\rho_c$ is the critical density of the universe, which is the density required for a flat universe; it is given by $\rho_c = \frac{3H^2}{8\pi G}$. Use $H_0 = 70\,\text{km/s/Mpc}$.

(h) (8 points) Assuming the baryon to photon ratio is fixed since the Big Bang:

   i. (5 points) Find an equation involving $T_{nuc}$ (the temperature at time $t = t_{nuc}$) and known constants.

   ii. (1 point) What temperature $T_{nuc}$ does $t_{nuc} = 200\,\text{s}$ correspond to?

   iii. (2 points) Verify that this temperature solves your equation in part (h)i.

(i) (3 points) The baryon to photon $\eta$ is a remarkably small number. One possibility is that the universe happens to prefer photons significantly over baryons. Another possibility is that a great number of quark-antiquark pairs were created in the early universe via pair production ($\gamma + \gamma \rightleftharpoons q + \bar{q}$), and a slight asymmetry of quarks over antiquarks produced a large number of photons during quark-antiquark annihilation, leaving over a small number of quarks to form into protons and neutrons. Find the quark-antiquark asymmetry

$$\delta_q \equiv \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} \ll 1$$

that would yield the baryon to photon ratio found in part (g).

**Solution:**

(a) i. When $k_B T \approx 0.8\,\text{MeV}$, we have

$$\begin{aligned} T &= \frac{0.8\,\text{MeV}}{k_B} \\ &= \frac{1.28 \cdot 10^{-13}\,\text{J}}{1.381 \cdot 10^{-23}\,\text{J/K}} \\ &= 9.28 \cdot 10^{9}\,\text{K}. \end{aligned}$$

Now, we have

$$t = \left(\frac{10^{10}}{T}\right)^2 = \boxed{1.16\,\text{s}}.$$

ii. Let $n_p$ be the number density of protons, and $n_n$ the number density of neutrons. Then, we have that the ratio $\frac{n_n}{n_p}$ is

$$\frac{\exp\left[-\frac{m_n c^2}{k_B T}\right]}{\exp\left[-\frac{m_p c^2}{k_B T}\right]} = \exp\left[-\frac{(m_n - m_p)c^2}{k_B T}\right].$$

Plugging in $m_nc^2 - m_pc^2 = 939.6 - 938.3 = 1.3\,\text{MeV}$ and $k_BT = 0.8\,\text{MeV}$, we get $\frac{n_n}{n_p} = \boxed{0.197}$.

(b) For every proton, there is initially 0.197 of a neutron. This decays according to $n_n = n_{n,0}\exp\left(-\frac{t}{\tau}\right)$. We have $0.197\exp\left(-\frac{200\,\text{s}}{886\,\text{s}}\right) = 0.157$, so the new ratio is $\frac{0.157}{1+0.197-0.157} = \boxed{0.151}$ (taking into account that the decayed neutrons turn into protons).

(c) i. Using the answer to part (b), for every 0.151 neutrons, there is 1 proton. Then, they can form $0.151/2 = 0.076$ helium-4 nuclei, and the remaining $1 - 0.151 = 0.849$ protons can form 0.849 hydrogen nuclei.
Thus, the ratio of helium nuclei to hydrogen nuclei is $\frac{0.075}{0.849} = \boxed{0.089}$.

ii. The mass ratio is $4 \cdot 0.089 = 0.356$, which gives a mass percentage of $\frac{0.356}{1+0.356} = \boxed{26.3\%}$

(d) Albert is $\boxed{\text{not correct}}$. Only free neutrons are unstable, and the vast majority of neutrons in the universe are bound up in nuclei, particularly helium-4. Furthermore, fusion in stars actually decreases the fraction of hydrogen, as explored in the following part.

(e) i. Since the electron and electron neutrino are of negligible mass, each reaction releases $4m_pc^2 - m_{He}c^2 = 4\cdot 938.3\,\text{MeV} - 3728.4\,\text{MeV} = 24.8\,\text{MeV}$. Hydrogen went from a density of $(1-0.263)\cdot 10^{-19}\,\text{kg/m}^3 = 7.37\times 10^{-20}\,\text{kg/m}^3$ to $0.70\cdot 10^{-19}\,\text{kg/m}^3 = 7\times 10^{-20}\,\text{kg/m}^3$, with a difference of $3.7 \times 10^{-21}\,\text{kg/m}^3$. This means that the energy released per cubic kiloparsec is

$$24.8\,\text{MeV}\cdot\frac{3.7\times 10^{-21}\,\text{kg/m}^3}{4\cdot 1.6726\times 10^{-27}\,\text{kg}}\cdot\left(\frac{3.086\times 10^{19}\,\text{m}}{1\,\text{kpc}}\right)^3\cdot\frac{1.6022\times 10^{-13}\,\text{J}}{1\,\text{MeV}} = \boxed{6.5\times 10^{52}\,\text{J/kpc}^3}.$$

ii. In seconds, 13.8 billion years is $13.8\times 10^9 \cdot 365.25\cdot 24\cdot 3600\,\text{s} = 4.35\times 10^{17}\,\text{s}$. Thus, the average power per cubic kiloparsec is $\frac{6.5\times 10^{52}\,\text{J/kpc}^3}{4.35\times 10^{17}\,\text{s}} = 1.50\times 10^{35}\,\text{W/kpc}^3$. In solar luminosities, this quantity is $\frac{1.50\times 10^{35}\,\text{W/kpc}^3}{3.85\times 10^{26}\,\text{W}/L_\odot} = \boxed{3.90\times 10^8\,L_\odot/\text{kpc}^3}$.

(f) From part (b), the ratio of neutrons to protons is 0.151. Thus, the proton to baryon ratio is $\frac{1}{1+0.151} = 0.869$, or $n_p = 0.869\,n_b$. By definition, $n_b = n_\gamma\eta = 0.243\eta\left(\frac{k_BT}{\hbar c}\right)^3$. We arrive at the expression $\boxed{n_p = 0.211\eta\left(\frac{k_BT}{\hbar c}\right)^3}$.

(g) Let us first find the present-day number density of photons. Simply plugging in $T = 2.725\,\text{K}$ into the expression given in part (f), we have

$$n_\gamma = 0.243\left(\frac{k_B\cdot 2.725\,\text{K}}{\hbar c}\right)^3 = 4.09\cdot 10^8\,\text{photons/m}^3.$$

To find the present-day number density of baryons, we first need to find the present-day critical density. Using $H_0 = 70\,\text{km/s/Mpc} = 2.3\cdot 10^{-18}\,\text{s}^{-1}$, we have $\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 9.2\cdot 10^{-27}\,\text{kg/m}^3$. Since $\Omega_{b,0} = 0.04$, $\rho_{b,0} = 0.04\cdot 9.2\cdot 10^{-27}\,\text{kg/m}^3 = 3.7\cdot 10^{-28}\,\text{kg/m}^3$. Finally, baryonic matter is composed of protons and neutrons; since $m_p \approx m_n$, we can divide by the mass of a proton and find the number density of baryons

$$n_b = \frac{3.7\cdot 10^{-28}\,\text{kg}}{\text{m}^3}\cdot\frac{1\,\text{baryon}}{1.6726\cdot 10^{-27}\,\text{kg}} = 0.22\,\text{baryons/m}^3.$$

The baryon to photon ratio is $\eta = \boxed{5.4\cdot 10^{-10}}$.

(h) i. Earlier, we defined $t_{nuc}$ to be the time at which half the neutrons fused into deuterium, or $\frac{n_D}{n_n} = 1$. Setting $\frac{n_D}{n_n} = 1$ and plugging in our expression for $n_p$ from part (f), we find

$$1 = 0.211\eta \left(\frac{k_B T}{\hbar c}\right)^3 \cdot 6 \left(\frac{m_n k_B T}{\pi \hbar^2}\right)^{-3/2} \exp\left(\frac{B_D}{k_B T}\right)$$

$$\boxed{1 \approx 7\eta \left(\frac{k_B T_{nuc}}{m_n c^2}\right)^{3/2} \exp\left(\frac{B_D}{k_B T_{nuc}}\right)}$$

ii. Using $T(t) \approx 10^{10}\,\mathrm{K}\left(\frac{t}{1\,\mathrm{s}}\right)^{-1/2}$, we get $T(t = 200\,\mathrm{s}) = \boxed{7 \cdot 10^8\ \mathrm{K}}$. The corresponding energy is $k_B T_{nuc} = 0.061\,\mathrm{MeV}$.

iii. Using $k_B T_{nuc} = 0.061\,\mathrm{MeV}$, we have

$$7\eta \left(\frac{k_B T_{nuc}}{m_n c^2}\right)^{3/2} \exp\left(\frac{B_D}{k_B T_{nuc}}\right)$$
$$= 7 \cdot 5.4 \cdot 10^{-10} \left(\frac{0.061\,\mathrm{MeV}}{939.6\,\mathrm{MeV}}\right)^{3/2} \exp\left(\frac{2.22\,\mathrm{MeV}}{0.061\,\mathrm{MeV}}\right) = 13$$

Due to the exponential, this expression is very sensitive to small changes in $k_B T$. Thus for this expression, an answer of 13 is roughly consistent with 1. The exact solution is $k_B T = 0.066$ MeV, which still corresponds to $t_{nuc} = 200$ s to the nearest significant figure.

(i) $2n_{\bar{q}}$ quarks are annihilated, producing $2n_{\bar{q}} \approx n_q + n_{\bar{q}}$ photons. $n_q - n_{\bar{q}}$ quarks are left over to form $(n_q - n_{\bar{q}})/3$ baryons. The resulting baryon to photon ratio is thus

$$\eta = \frac{(n_q - n_{\bar{q}})/3}{n_q + n_{\bar{q}}} = \frac{1}{3}\delta_q.$$

Using $\eta = 5.4 \cdot 10^{-10}$, we find a quark-antiquark asymmetry of $\boxed{\delta_q = 1.6 \cdot 10^{-9}}$; there was one extra quark in 800 million quark-antiquark pairs.

2. (**35 points**) In 2020, during the day of the winter solstice for the Northern hemisphere, Jupiter and Saturn were at their minimum angular separation (approximately 6.11') during the Great Conjunction.

(a) (6 points) Consider a system with three planets in circular, concentric, and coplanar orbits around a star. Suppose that the three planets and the star are initially aligned. Will they necessarily align again after this moment? Prove your answer with quantitative arguments. Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(b) (4 points) Suppose that there were N planets instead of three in the system from item A. N is an integer greater than 3. If the orbits were still circular, concentric, and coplanar, and the planets and star were all initially aligned, would they necessarily align again afterwards? Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(c) (8 points) In the system from (a), if the three planets were not initially aligned with respect to the star, would they necessarily be perfectly aligned at some point? Again, use quantitative arguments to prove your answer.

(d) (4 points) Suppose that you are an astronomer who wants to use a telescope to observe the conjunction. Since you are a very skilled astronomer, you are going to build your own telescope. The only basic requirement you want to meet is that your telescope must be able to resolve the planets at the minimum separation during the conjunction. Calculate the value of all parameters of your telescope that are relevant for this goal. Do not try to calculate the values of any parameters that are not related to this requirement.

(e) (8 points) Calculate the total apparent magnitude of the planets together in the conjunction. Assume that the observers see Jupiter and Saturn as a single point in the sky, but Saturn is not covered (totally or partially) by Jupiter. For this item, neglect the atmospheric extinction, consider that the planets reflect isotropically, and consider that the albedos of both Jupiter and Saturn are equal to one. Also, in order to make the calculations simpler, assume that both Jupiter and Saturn were almost in opposition with respect to the Earth (even though this was not the case for this conjunction).

(f) (5 points) Calculate the difference in the magnitude of the conjunction at the zenith and at a zenith distance of $15^o$. Assume that the zenith optical depth of Earth's atmosphere for visible light is 0.50.

- Mean orbital radius of Jupiter: 5.2 AU
- Mean orbital radius of Saturn: 9.5 AU
- Radius of Jupiter: $7.1492 \times 10^7$ meters
- Radius of Saturn: $5.8232 \times 10^7$ meters
- Apparent magnitude of the Sun: $-26.74$
- Central wavelength of visible light: 550 nm

**Solution:**

(a) It is possible to write the following expression for the synodic period between the first two planets ($S_{1,2}$, in which planet 1 is the closest to the star, and $T_n$ represents the sidereal period of the $n^{th}$ planet):

$$\frac{1}{S_{1,2}} = \frac{1}{T_1} - \frac{1}{T_2}$$
$$S_{1,2} = \frac{T_1 T_2}{T_2 - T_1}$$

Likewise, the synodic period between planets 1 and 3 ($S_{1,3}$) is the following:

$$S_{1,3} = \frac{T_1 T_3}{T_3 - T_1}.$$

It is important to notice that both synodic periods are rational numbers. All sidereal periods are rational numbers, and by definition, the subtraction or division of two rational numbers must result in a rational number, so both $S_{1,2}$ and $S_{1,3}$ are rational.

If we assume that the planets are aligned at $t = 0$, planets 1 and 2 must be aligned at all instants $t = m \cdot S_{1,2}$, in which $m \in \mathbb{N}$. Likewise, planets 1 and 3 must be aligned at all instants $t = n \cdot S_{1,3}$, in which $n \in \mathbb{N}$. Therefore, the following condition must be met for a triple alignment:

$$m \cdot S_{1,2} = n \cdot S_{1,3}$$
$$\frac{m}{n} = \frac{S_{1,3}}{S_{1,2}}$$
$$\frac{m}{n} = r.$$

Since $r$ is the ratio between two rational numbers, it must also be a rational number. By definition, every rational number might be expressed as the ratio between two integers. In this case, since $r$ must be positive, the two integers must have the same sign. Therefore, this equation has solutions in which $m, n \in \mathbb{N}$, so the planets will align periodically after $t = 0$.

(b) It is possible to expand the alignment condition from item A to more planets:

$$m_1 \cdot S_{1,2} = m_2 \cdot S_{2,3} = ... = m_n \cdot S_{n,n+1}.$$

In this equation, all $m_n$ terms are natural numbers greater than zero.

Again, it is possible to use the definition of a rational number to solve this problem. As demonstrated on item A, if the sidereal periods are rational numbers, the synodic periods must be rational as well. Since a rational number might be expressed as a ratio between two integers, the product $m_n \cdot S_{n,n+1}$ is an integer for certain values of $m_n$. Therefore, there are values of $m_n$ for which all $m_n \cdot S_{n,n+1}$ terms in the alignment equality are natural numbers. By definition, any set of natural numbers has a least common multiple (LCM), and an infinite number of common multiples. Therefore, it is possible to multiply all $m_n \cdot S_{n,n+1}$ terms by an integer factor to obtain the LCM. If all terms are equal to the LCM, the alignment equality is true, which proves that all planets are aligned at an instant later than $t = 0$.

(c) In order to obtain an expression for the instants $t_{1,2}$ in which the first two planets are aligned, it is possible to write the following formula, in which m $\in \mathbb{N}$ and $\theta_{i,n}$ corresponds to the initial angular position of the $n^{th}$ planet:

$$\theta_{i,1} + \frac{2\pi}{T_1} t_{1,2} + 2\pi m = \theta_{i,2} + \frac{2\pi}{T_2} t_{1,2}$$

$$t_{1,2} \left( \frac{1}{T_1} - \frac{1}{T_2} \right) = \left( \frac{\theta_{i,2} - \theta_{i,1}}{2\pi} - m \right)$$

$$t_{1,2} = \frac{1}{S_{1,2}} \left( \frac{\theta_{i,2} - \theta_{i,1}}{2\pi} - m \right)$$

Likewise, the formula for the alignment between planets 1 and 3 will be the following:

$$t_{1,3} = \frac{1}{S_{1,3}} \left( \frac{\theta_{i,3} - \theta_{i,1}}{2\pi} - n \right).$$

In this formula, $n$ is a natural number.

Therefore, it is possible to write the following equality for the alignment between three planets:

$$\frac{1}{S_{1,2}} \left( \frac{\theta_{i,2} - \theta_{i,1}}{2\pi} - m \right) = \frac{1}{S_{1,3}} \left( \frac{\theta_{i,3} - \theta_{i,1}}{2\pi} - n \right)$$

In order to simplify the formula, it is possible to group a few variables together:

$$\alpha = \frac{S_{1,3}}{S_{1,2}}$$

$$\beta = \frac{\theta_{i,2} - \theta_{i,1}}{2\pi}$$

$$\gamma = \frac{\theta_{i,3} - \theta_{i,1}}{2\pi}$$

Since $\alpha$, $\beta$, and $\gamma$ are the result of consecutive subtractions and divisions of rational numbers, they are also rational numbers. However, they are not necessarily rational.

It is possible to rewrite the formula using $\alpha$, $\beta$, and $\gamma$:

$$\alpha(\beta - m) = (\gamma - n)$$

$$\alpha m - n = \alpha\beta - \gamma$$

It is possible to define a new variable $\delta$ to simplify the expression even more:

$$\alpha m - n = \delta$$

From this expression, it is clear that there are values of $\alpha$ and $\delta$ for which there are no solutions in which both $m$ and $n$ are natural numbers.

The easiest way to demonstrate this is with a simple counterexample. Suppose that $\alpha = 2$ and $\delta = 3.5$.

$$2m - n = 3.5$$

If both 2n and m are integers, the result of the subtraction should necessarily be an integer. However, since 3.5 is not an integer, there are no solutions for which both m and n are integers. Therefore, if the three planets are not initially aligned, they might never have a triple conjunction.

It is important to highlight that $\alpha$ is a function only of the planets' periods, but $\delta$ is a function of both the periods and the initial angular positions. Therefore, the method of choosing arbitrary values for a counterexample is valid.

**Note:** For the first three items of this question, students were allowed to assume that the periods of the planets are rational numbers.

However, the period of a body in a circular motion is not necessarily a rational number. It must be a real number, but it might also be an irrational number. For instance, consider a circular movement with an angular velocity of 1 rad/s. In this case, the period would be equal to $2\pi$, which is an irrational number.

The reason why this question assumes that the periods are rational numbers is that it is impossible to measure periods with an infinite number of significant figures. On the aforementioned example, although $2\pi$ is an irrational number, 6.28319 is rational. Since we can only measure periods with a limited number of significant figures, even our most precise measurement for the period of any planet will still be a rational number.

It is also important to notice that in a real life situation, it is essential to consider that planets are not point particles and take into account the radius of each planet. Consider a hypothetical system with three planets with periods of 0.5 year, 1 year, and 2.00000000000000000000001 years. It is clear that if the planets are initially aligned, they will be pretty much aligned again in two years (considering that the third period is equal to approximately 2 years). However, because the third period is not exactly 2 years, the time interval until the next perfect alignment is very long. In this case, it makes much more sense to consider that the period between the alignments is equal to two years, not to a very large number.

(d) In this the, the angular resolution must be less of equal to 6.11'. Besides the wavelength, the diameter is the only parameter that affects the angular resolution. It is possible to use the following formula to calculate the diameter:

$$\theta = 1.22\frac{\lambda}{D}$$

$$D = 1.22\frac{\lambda}{\theta}$$

Since $\theta \leqslant 6.11'$ and visible light is centered at 550 nm:

$$D \geqslant 1.22\frac{5.50 \times 10^{-7}}{6.11\pi/(60 \times 180)}$$

$$D \geqslant 3.78 \times 10^{-4} m$$

$\therefore$ The diameter of the telescope must be greater or equal to $3.78 \times 10^{-4}$ m. In other words, basically any telescope you could possibly build will meet this requirement.

(e) Solar flux that arrives at Jupiter and Saturn:

$$F_J = \frac{L_\odot}{4\pi r_J^2}$$

$$F_S = \frac{L_\odot}{4\pi r_S^2}$$

Flux from Jupiter and Saturn that arrives at the Earth:

$$F_{\oplus 1} = \frac{F_J \pi R_J^2}{4\pi(r_J - r_\oplus)^2} + \frac{F_S \pi R_S^2}{4\pi(r_S - r_\oplus)^2}$$

$$F_{\oplus 1} = \frac{L_\odot R_J^2}{16\pi r_J^2 (r_J - r_\oplus)^2} + \frac{L_\odot R_S^2}{16\pi r_S^2 (r_S - r_\oplus)^2}$$

$$F_{\oplus 1} = \frac{L_\odot}{16\pi}\left(\frac{R_J^2}{r_J^2 (r_J - r_\oplus)^2} + \frac{R_S^2}{r_S^2 (r_S - r_\oplus)^2}\right)$$

Solar flux that arrives at the Earth:

$$F_{\oplus 2} = \frac{L_\odot}{4\pi r_\oplus^2}$$

Ratio between the fluxes:

$$\frac{F_{\oplus 1}}{F_{\oplus 2}} = \frac{r_\oplus^2}{4}\left(\frac{R_J^2}{r_J^2 (r_J - r_\oplus)^2} + \frac{R_S^2}{r_S^2 (r_S - r_\oplus)^2}\right)$$

$$\frac{F_{\oplus 1}}{F_{\oplus 2}} = \frac{(1.496 \times 10^{11})^2}{4}\left(\frac{(7.1492 \times 10^7)^2}{5.2^2 \times 4.2^2 \times (1.496 \times 10^{11})^4} + \frac{(5.8232 \times 10^7)^2}{9.5^2 \times 8.5^2 \times (1.496 \times 10^{11})^4}\right)$$

$$\frac{F_{\oplus 1}}{F_{\oplus 2}} = 1.2551 \times 10^{-10}$$

Using Pogson's Law:

$$\begin{aligned} m_{\text{Conjunction}} - m_\odot &= -2.5 \times \log\left(\frac{F_{\oplus 1}}{F_{\oplus 2}}\right) \\ m_{\text{Conjunction}} &= -2.5 \times \log(1.2551 \times 10^{-10}) - 26.74 \\ m_{\text{Conjunction}} &= -1.99 \end{aligned}$$

∴ The apparent magnitude of the great conjunction is equal to −1.99.

**Note:** While students were supposed to assume that an observer would see Jupiter and Saturn as a single point in the sky, the minimum separation of 6.11' corresponds to about 1/5 of the Moon's angular diameter, so the human eye can easily resolve this angular separation.

(f) The optical depth ($\tau$) is defined by the following expression, in which $\kappa$ is the opacity coefficient, $\rho$ is the density of the medium, and s is the distance travelled in the medium:

$$\tau = \kappa\rho s$$

Using the flat atmosphere approximation, which works well for small zenith distances, it is possible to calculate the optical depth of the Earth's atmosphere at a zenith distance of $15^o$ ($\tau_{15}$). The only factor that varies in this case is the distance travelled, which is equal to $d_z \times sec(15^o)$, in which $d_z$ is the distance travelled for a zenith distance of $0^o$.

Therefore, is is possible to use the zenith optical depth $\tau_z$ to calculate $\tau_{15}$:

$$\tau_{15} = \tau_z \times sec(15^o)$$
$$\tau_{15} = \tau_z \times sec(15^o)$$
$$\tau_{15} = 0.5 \times sec(15^o)$$
$$\tau_{15} = 0.5176$$

Considering that the ratio between the flux outside the atmosphere and the flux after atmospheric extinction is by definition the exponential function of the optical depth, it is possible to calculate the ratio between the fluxes for the zenith and for a zenith distance of $15^o$:

$$\begin{aligned} \frac{F_{15}}{F_z} &= \frac{F_0 \cdot e^{-\tau_{15}}}{F_0 \cdot e^{-\tau_z}} \\ &= e^{\tau_z - \tau_{15}} \\ &= e^{0.5-0.5176} \\ &= 0.9825. \end{aligned}$$

Now, it is possible to use Pogson's Law to calculate the difference in magnitude:

$$\begin{aligned} \Delta m &= -2.5 \cdot \log\left(\frac{F_{15}}{F_z}\right) \\ &= -2.5 \cdot \log(0.9825) \\ &= 1.92 \times 10^{-2}. \end{aligned}$$

∴ The difference in magnitude between the Great Conjunction seen at a zenith distance of $15^\circ$ and the Great Conjunction seen at the zenith corresponds to $1.92 \times 10^{-2}$.