4 Long Questions

1. (**40 points**) In the very early universe, everything is in thermodynamic equilibrium and particles are freely created, destroyed, and converted between each other due to the high temperature. In one such process, the reaction converting between neutrons and protons happens at a very high rate. In thermal equilibrium, the relative number density of particle species is given approximately by the Boltzmann factor:

$$n_i \propto \exp\left[-\frac{E_i}{k_B T}\right],$$

where $E_i = m_i c^2$ is the rest energy. Additionally, the temperature during the radiation-dominated early universe is given by $T(t) \approx 10^{10}\,\mathrm{K}\left(\frac{t}{1\,\mathrm{s}}\right)^{-1/2}$, where t is the time since the Big Bang.

   (a) (4 points) At a temperature where $k_B T \approx 0.8\,\mathrm{MeV}$, known as the *freeze-out* temperature, the neutrino interactions essentially stop, preventing further conversion between protons and neutrons.

      i. (2 points) About how long after the Big Bang did this occur?

      ii. (2 points) At the freeze-out temperature, what was the equilibrium ratio of the number density of neutrons to that of protons?

   (b) (3 points) Free neutrons are unstable, and decay into protons with a characteristic decay time of $\tau = 886\,\mathrm{s}$ (the time for which the number of neutrons drops to $1/e$ of the original amount). Given that helium nuclei only formed $t_{nuc} = 200\,\mathrm{s}$ after freezing out, what was the ratio of the number density of neutrons to that of protons when the helium nuclei formed?

   (c) (7 points) While trace amounts of several small nuclei were formed during Big Bang Nucleosynthesis (BBN), assume that all neutrons go into forming helium-4.

      i. (5 points) After the helium nuclei formed, what was the ratio of the number of helium-4 nuclei to the number of hydrogen nuclei?

      ii. (2 points) Approximating the mass of helium-4 as 4 times that of H (for this part only), what fraction of baryonic mass in the universe is helium?

If you weren't able to solve part (c), assume reasonable values for the initial mass fractions of hydrogen and helium for future parts.

   (d) (2 points) Albert the Astronomer claims that in older galaxies, the mass fraction of hydrogen should gradually be increasing, as neutrons slowly continue to decay into protons. Is his claim correct? If not, explain.

   (e) (7 points) Suppose a certain region of a galaxy has a density of $10^{-19}\,\mathrm{kg/m^3}$ and is composed of 70% hydrogen and 30% helium-4 by mass (ignore any heavier elements). Because the region is gravitationally bound, this density doesn't change significantly with the expansion of the universe; approximate it as constant. Assume hydrogen is converted into helium by the fusion reaction:

$$4\,{}^{1}\mathrm{H}^{+} + 2\mathrm{e}^{-} \rightarrow {}^{4}\mathrm{He}^{2+} + 2\nu_e,$$

   where the electron e^{-} and electron neutrino ν_e are of negligible mass. ${}^{4}\mathrm{He}$ has a mass of $m_{He} = 3728.4\,\mathrm{MeV/c^2}$

      i. (4 points) Over the entire time since BBN, how much energy does this process release per cubic kiloparsec? Give your answer in joules per cubic kiloparsec.

      ii. (3 points) Assuming the age of the universe is 13.8 billion years, calculate the average luminosity density in solar luminosities per cubic kiloparsec.

Let's go back and explore how we arrived at the number $t_{nuc} \approx 200\,\mathrm{s}$, the time at which Big Bang nucleosynthesis began. Let's define t_{nuc} as the time at which half the neutrons fused with protons into deuterium (^{2}H), as deuterium fusion is the first step in BBN. From the Maxwell-Boltzmann equation, the relative abundances of deuterium, protons and neutrons is given by

$$\frac{n_D}{n_p n_n} = 6\left(\frac{m_n k_B T}{\pi \hbar^2}\right)^{-3/2} \exp\left(\frac{B_D}{k_B T}\right),$$

where $B_D = (m_p + m_n - m_D)\,c^2 = 2.22\,\text{MeV}$ is the energy released in a deuterium fusion reaction.

(f) (3 points) The number density of photons is given by $n_\gamma = 0.243\left(\frac{k_B T}{\hbar c}\right)^3$. Find an expression for the number density of protons n_p in terms of the temperature T and the baryon to photon ratio η. You may use your answer to part (b).

(g) (3 points) Find the present-day baryon to photon ratio. The CMB temperature is 2.725 K, and the present-day density parameter for baryonic matter is $\Omega_{b,0} = \frac{\rho_{b,0}}{\rho_{c,0}} = 0.04$. ρ_c is the critical density of the universe, which is the density required for a flat universe; it is given by $\rho_c = \frac{3H^2}{8\pi G}$. Use $H_0 = 70\,\text{km/s/Mpc}$.

(h) (8 points) Assuming the baryon to photon ratio is fixed since the Big Bang:

  i. (5 points) Find an equation involving T_{nuc} (the temperature at time $t = t_{nuc}$) and known constants.

  ii. (1 point) What temperature T_{nuc} does $t_{nuc} = 200\,\text{s}$ correspond to?

  iii. (2 points) Verify that this temperature solves your equation in part (h)i.

(i) (3 points) The baryon to photon η is a remarkably small number. One possibility is that the universe happens to prefer photons significantly over baryons. Another possibility is that a great number of quark-antiquark pairs were created in the early universe via pair production ($\gamma + \gamma \rightleftharpoons q + \bar{q}$), and a slight asymmetry of quarks over antiquarks produced a large number of photons during quark-antiquark annihilation, leaving over a small number of quarks to form into protons and neutrons. Find the quark-antiquark asymmetry

$$\delta_q \equiv \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} \ll 1$$

that would yield the baryon to photon ratio found in part (g).

2. (**35 points**) In 2020, during the day of the winter solstice for the Northern hemisphere, Jupiter and Saturn were at their minimum angular separation (approximately 6.11') during the Great Conjunction.

(a) (6 points) Consider a system with three planets in circular, concentric, and coplanar orbits around a star. Suppose that the three planets and the star are initially aligned. Will they necessarily align again after this moment? Prove your answer with quantitative arguments. Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(b) (4 points) Suppose that there were N planets instead of three in the system from item A. N is an integer greater than 3. If the orbits were still circular, concentric, and coplanar, and the planets and star were all initially aligned, would they necessarily align again afterwards? Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(c) (8 points) In the system from (a), if the three planets were not initially aligned with respect to the star, would they necessarily be perfectly aligned at some point? Again, use quantitative arguments to prove your answer.

(d) (4 points) Suppose that you are an astronomer who wants to use a telescope to observe the conjunction. Since you are a very skilled astronomer, you are going to build your own telescope. The only basic requirement you want to meet is that your telescope must be able to resolve the planets at the minimum separation during the conjunction. Calculate the value of all parameters of your telescope that are relevant for this goal. Do not try to calculate the values of any parameters that are not related to this requirement.

(e) (8 points) Calculate the total apparent magnitude of the planets together in the conjunction. Assume that the observers see Jupiter and Saturn as a single point in the sky, but Saturn is not covered (totally or partially) by Jupiter. For this item, neglect the atmospheric extinction, consider that the planets reflect isotropically, and consider that the albedos of both Jupiter and Saturn are equal to one. Also, in order to make the calculations simpler, assume that both Jupiter and Saturn were almost in opposition with respect to the Earth (even though this was not the case for this conjunction).

(f) (5 points) Calculate the difference in the magnitude of the conjunction at the zenith and at a zenith distance of 15^o. Assume that the zenith optical depth of Earth's atmosphere for visible light is 0.50.

- Mean orbital radius of Jupiter: 5.2 AU
- Mean orbital radius of Saturn: 9.5 AU
- Radius of Jupiter: 7.1492×10^7 meters
- Radius of Saturn: 5.8232×10^7 meters
- Apparent magnitude of the Sun: -26.74
- Central wavelength of visible light: 550 nm