(T1) **True or False**

Determine if each of the following statements is True or False. In the Summary Answersheet, tick the correct answer (TRUE / FALSE) for each statement. No justifications are necessary for this question.

(T1.1) In a photograph of the clear sky on a Full Moon night with a sufficiently long exposure, the colour of the sky would appear blue as in daytime. **2**

Solution:

T

The colour of the clear sky during night is the same as during daytime, since the spectrum of sunlight reflected by the Moon is almost the same as the spectrum of sunlight. Only the intensity is lower. **2.0**

(T1.2) An astronomer at Bhubaneswar marks the position of the Sun on the sky at 05:00 UT every day of the year. If the Earth's axis were perpendicular to its orbital plane, these positions would trace an arc of a great circle. **2**

Solution:

T

If the Earth's axis were perpendicular to its orbital plane, the celestial equator will coincide with ecliptic and the Sun will remain along the celestial equator every day. However, as the Earth's orbit is elliptical, the true sun would still lead or lag mean sun by a few minutes on different days of year. **2.0**

(T1.3) If the orbital period of a certain minor body around the Sun in the ecliptic plane is less than the orbital period of Uranus, then its orbit must necessarily be fully inside the orbit of Uranus. **2**

Solution:

F

The semi-major axis of the orbit of the body will be less than that of Uranus. However the minor body's orbit may have a high eccentricity, in which case it may go outside that of Uranus. **2.0**

(T1.4) The centre of mass of the solar system is inside the Sun at all times. **2**

Solution:

F

The centre of mass of Sun-Jupiter pair is just outside the Sun. Thus, if all gas giants are on same side of the Sun, the centre of mass of Solar system is definitely outside the Sun. **2.0**

(T1.5) A photon is moving in free space. As the Universe expands, its momentum decreases. **2**

Solution:

T

For photons the wavelength increases when the Universe expands. **2.0**

(T2) **Gases on Titan** **10**

Gas particles in a planetary atmosphere have a wide distribution of speeds. If the r.m.s. (root mean square) thermal speed of particles of a particular gas exceeds 1/6 of the escape speed, then most of that gas will escape from the planet. What is the minimum atomic weight (relative atomic mass), A_{min}, of an ideal monatomic gas so that it remains in the atmosphere of Titan?

Given, mass of Titan $M_{\text{T}} = 1.23 \times 10^{23}\,\text{kg}$, radius of Titan $R_{\text{T}} = 2575\,\text{km}$, surface temperature of Titan $T_{\text{T}} = 93.7\,\text{K}$.

Solution:

As the gas is monatomic,

$$\frac{3}{2}k_{\text{B}}T_{\text{T}} \approx \frac{1}{2}m_{\text{g}}v_{\text{rms}}^2$$

$$3k_{\text{B}}T_{\text{T}} \approx \frac{M_{\text{g}}}{N_{\text{A}}}v_{\text{rms}}^2$$

$$\therefore v_{\text{rms}} \approx \sqrt{\frac{3k_{\text{B}}N_{\text{A}}T_{\text{T}}}{M_{\text{g}}}} \qquad \textbf{4.0}$$

50% **deduction if** $3/2$ **pre-factor is not used and** $1/2$ **or** 1 **are used instead.**
Full credit if students writes the relation for v_{rms} **directly.**

To remain in atmosphere,

$$v_{\text{rms}} < \frac{v_{\text{esc}}}{6} = \frac{1}{6}\sqrt{\frac{2GM_{\text{T}}}{R_{\text{T}}}}$$

$$\sqrt{\frac{3k_{\text{B}}N_{\text{A}}T_{\text{T}}}{M_{\text{g}}}} < \sqrt{\frac{GM_{\text{T}}}{18R_{\text{T}}}}$$

$$\therefore M_{\text{g}} > \frac{54k_{\text{B}}N_{\text{A}}T_{\text{T}}R_{\text{T}}}{GM_{\text{T}}} \qquad \textbf{4.0}$$

$$> \frac{54 \times 1.381 \times 10^{-23} \times 6.022 \times 10^{23} \times 93.7 \times 2.575 \times 10^{6}}{6.6741 \times 10^{-11} \times 1.23 \times 10^{23}}\,\text{g}$$

$$> 13.2\,\text{g} \qquad \textbf{1.5}$$

Thus, all gases with atomic weight more than $\boxed{A_{\text{min}} = 13.2}$ will be retained in the atmosphere of Titan. **0.5**

Half mark for understanding that atomic mass has no units.

Alternative solution

$$\frac{3}{2}k_{\text{B}}T_{\text{T}} \approx \frac{1}{2}m_{\text{g}}v_{\text{rms}}^2$$

$$\therefore v_{\text{rms}} \approx \sqrt{\frac{3k_{\text{B}}T_{\text{T}}}{m_{\text{g}}}} \qquad \textbf{4.0}$$

$$\therefore m_{\text{g}} > \frac{54k_{\text{B}}T_{\text{T}}R_{\text{T}}}{GM_{\text{T}}} \qquad \textbf{4.0}$$

$$m_{\text{g}} > 2.19 \times 10^{-26}\,\text{kg} \qquad \textbf{1.0}$$

$$\therefore A_{\text{min}} = \frac{m_{\text{g}}}{\text{atomic mass unit}} = \frac{2.19 \times 10^{-26}\,\text{kg}}{1.66 \times 10^{-27}\,\text{kg}}$$

$\boxed{A_{\min} = 13.2}$ **1.0**

Answers between 13.0 and 13.4 are acceptable with full credit.

(T3) **Early Universe**

Cosmological models indicate that radiation energy density, $\rho_{\rm r}$, in the Universe is proportional to $(1+z)^4$, and the matter energy density, $\rho_{\rm m}$, is proportional to $(1+z)^3$, where z is the redshift. The dimensionless density parameter, Ω, is given as $\Omega = \rho/\rho_{\rm c}$, where $\rho_{\rm c}$ is the critical energy density of the Universe. In the present Universe, the density parameters corresponding to radiation and matter, are $\Omega_{\rm r_0} = 10^{-4}$ and $\Omega_{\rm m_0} = 0.3$, respectively.

(T3.1) Calculate the redshift, $z_{\rm e}$, at which radiation and matter energy densities were equal. **3**

Solution:

$$\frac{\rho_{\rm m_0}/\rho_{\rm c}}{\rho_{\rm r_0}/\rho_{\rm c}} = \frac{\Omega_{\rm m_0}}{\Omega_{\rm r_0}} = \frac{0.3}{10^{-4}} = 3000$$

At $z_{\rm e}$, both matter density and radiation density were equal.

$$\rho_{\rm r} = \rho_{\rm m}$$

$$\therefore \rho_{\rm r_0}(1+z_{\rm e})^4 = \rho_{\rm m_0}(1+z_{\rm e})^3$$

$$1 + z_{\rm e} = \frac{\rho_{\rm m_0}}{\rho_{\rm r_0}} = 3000 \quad \textbf{2.0}$$

$$\therefore z_{\rm e} \simeq \boxed{3000} \quad \textbf{1.0}$$

Only $z_{\rm e} = 2999$ and $z_{\rm e} = 3000$ are acceptable answers.

(T3.2) Assuming that the radiation from the early Universe has a blackbody spectrum with a temperature of 2.732 K, estimate the temperature, $T_{\rm e}$, of the radiation at redshift $z_{\rm e}$. **4**

Solution:

As the Universe behaves like an ideal black body, the radiation density will be proportional to the fourth power of the temperature (Stefan's law).

$$\left(\frac{T_{\rm e}}{T_0}\right)^4 = \frac{\rho_{\rm r_e}}{\rho_{\rm r_0}} = \frac{\rho_{\rm r_0}(1+z_{\rm e})^4}{\rho_{\rm r_0}} \quad \textbf{2.0}$$

$$\left(\frac{T_{\rm e}}{2.732}\right)^4 = (1+z_{\rm e})^4 \quad \textbf{1.0}$$

$$\frac{T_{\rm e}}{2.732} = 1 + z_{\rm e} = 3000$$

$$T_{\rm e} = 3000 \times 2.732$$

$$T_{\rm e} = \boxed{8200\,{\rm K}} \quad \textbf{1.0}$$

$8100 \le T_{\rm e} \le 8200$ **gives 1.0;** $8200 < T_{\rm e} \le 9000$ **gives 0.5; else 0.**

(T3.3) Estimate the typical photon energy, E_ν (in eV), of the radiation as emitted at redshift $z_{\rm e}$. **3**

Solution:

Wien's law:

$$\lambda_{\max} = \frac{0.002898\,\mathrm{m\,K}}{T_e}$$

$$= \frac{0.002898}{8200}\,\mathrm{m} = 354\,\mathrm{nm} \quad \textbf{1.0}$$

$$E_\nu = \frac{hc}{\lambda_{\max}}$$

$$= \frac{6.62\times10^{-34}\times3\times10^{8}}{354\times10^{-9}}\,\mathrm{J} = \frac{5.62\times10^{-19}}{1.602\times10^{-19}}\,\mathrm{eV} \quad \textbf{1.0}$$

$$E_\nu = \boxed{3.5\,\mathrm{eV}} \quad \textbf{1.0}$$

Alternative solution:

$$E_\nu = k_\mathrm{B}T_\mathrm{e}$$

$$= 1.38\times10^{-23}\times8200\,\mathrm{J} = \frac{1.13\times10^{-19}}{1.602\times10^{-19}}\,\mathrm{eV} \quad \textbf{2.0}$$

$$E_\nu = \boxed{0.71\,\mathrm{eV}} \quad \textbf{1.0}$$

Use of either Wien's law or $E = k_\mathbf{B}T$ gets full credit. $E_\nu = 3k_\mathbf{B}T/2$ or similar gets no credit. Answers with $E_\nu = 3k_\mathbf{B}T$ or $E_\nu = 2.7k_\mathbf{B}T$ also get full credit.

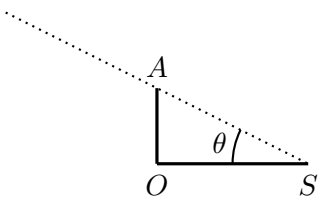
(T4) **Shadows** **10**

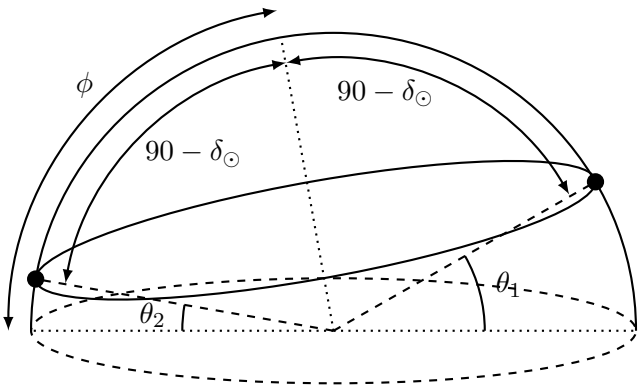
An observer in the northern hemisphere noticed that the length of the shortest shadow of a 1.000 m vertical stick on a day was 1.732 m. On the same day, the length of the longest shadow of the same vertical stick was measured to be 5.671 m.

Find the latitude, ϕ, of the observer and declination of the Sun, $\delta_\odot$, on that day. Assume the Sun to be a point source and ignore atmospheric refraction.

Solution:

As the longest shadow of the Sun on the given day is of finite length, the Sun is circumpolar for this observer on this day. **2.0**





In the figure above, the left panel shows the shadow OS formed by stick OA (of length 1.000 m), and the right panel shows the Sun's location in two cases.

For an altitude θ of the Sun,

$$\tan\theta = \frac{OA}{OS} = \frac{1.000\,\text{m}}{OS}$$

$$\therefore \cot\theta = OS\,(\text{in metres})$$ **1.0**

Let θ_1 and θ_2 be altitude in two extreme cases.

$$\theta_1 = 180^\circ - \phi - (90^\circ - \delta_\odot) = 90^\circ - \phi + \delta_\odot$$ **1.0**

$$\cot(90^\circ - \phi + \delta_\odot) = 1.732$$
$$\therefore \tan(\phi - \delta_\odot) = 1.732$$
$$\phi - \delta_\odot = \tan^{-1}(1.732) = 60^\circ = 1.047\,\text{rad}$$ **1.5**

$$\theta_2 = \phi - (90^\circ - \delta_\odot) = \phi - 90^\circ + \delta_\odot$$ **1.0**

$$\cot(\phi - 90^\circ + \delta_\odot) = 5.671$$
$$\therefore \tan(\phi - 90^\circ + \delta_\odot) = \frac{1}{5.671}$$
$$\phi + \delta_\odot = \tan^{-1}\left(\frac{1}{5.671}\right) + 90^\circ = 100^\circ = 1.745\,\text{rad}$$ **1.5**

Solving,

$$\boxed{\phi = 80^\circ} = 1.396\,\text{rad}$$ **1.0**

$$\boxed{\delta_\odot = 20^\circ} = 0.349\,\text{rad}$$ **1.0**

Given high accuracy of shadow length, only $\pm 0.5^\circ$ is allowed.

One can also solve the question by manipulating $\tan(\phi - \delta_\odot)$ and $\tan(\phi + \delta_\odot)$, to get $\tan(\phi)$ and $\tan(\delta_\odot)$.

(T5) **GMRT beam transit** **10**

Giant Metrewave Radio Telescope (GMRT), one of the world's largest radio telescopes at metre wavelengths, is located in western India (latitude: 19°6′ N, longitude: 74°3′ E). GMRT consists of 30 dish antennas, each with a diameter of 45.0 m. A single dish of GMRT was held fixed with its axis pointing at a zenith angle of 39°42′ along the northern meridian such that a radio point source would pass along a diameter of the beam, when it is transiting the meridian.

What is the duration T_{transit} for which this source would be within the FWHM (full width at half maximum) of the beam of a single GMRT dish observing at 200 MHz?

Hint: The FWHM size of the beam of a radio dish operating at a given frequency corresponds to the angular resolution of the dish. Assume uniform illumination.

Solution:

As the dish is pointed towards northern meridian at zenith angle of 39.7°, altitude of the centre of the beam is

$$a = 90.00^\circ - z = 90.00^\circ - 39.70^\circ = 50.30^\circ$$ **1.0**

Thus, declination of the source should be,

$$\delta = 90.00 - a + \phi = 90.00^\circ - 50.30^\circ + 19.10^\circ = 58.80^\circ$$ **2.0**

Declination = ZA + Latitude also gets full credit.

FWHM beam size (for uniform illumination) will be given by

$$\theta = \frac{1.22\lambda}{D}$$ **1.5**

$$= \frac{1.22c}{D\nu} = \frac{1.22 \times 2.998 \times 10^8}{45.0 \times 2 \times 10^8}$$

$$= 0.0406\,\text{rad}$$

$$\theta = 2.33^\circ$$ **1.5**

$$T_{\text{transit}} = \frac{\theta \times 3.99\,\text{min}}{\cos\delta}$$ **3.0**

$$= \frac{2.33 \times 3.99\,\text{min}}{\cos 58.8}$$

$$\boxed{T_{\text{transit}} = 17.9\,\text{min}}$$ **1.0**

- **Use of 4 min per degree is also acceptable.**
- **Missing $\cos\delta$ gets a penalty of 2.0.**

(T6) **Cepheid Pulsation**

The star β-Doradus is a Cepheid variable star with a pulsation period of 9.84 days. We make a simplifying assumption that the star is brightest when it is most contracted (radius being R_1) and it is faintest when it is most expanded (radius being R_2). For simplicity, assume that the star maintains its spherical shape and behaves as a perfect black body at every instant during the entire cycle. The bolometric magnitude of the star varies from 3.46 to 4.08. From Doppler measurements, we know that during pulsation the stellar surface expands or contracts at an average radial speed of $12.8\,\text{km}\,\text{s}^{-1}$. Over the period of pulsation, the peak of thermal radiation (intrinsic) of the star varies from 531.0 nm to 649.1 nm.

(T6.1) Find the ratio of radii of the star in its most contracted and most expanded states (R_1/R_2). **7**

Solution:

We first find flux ratio and then use Stefan's law to compare the fluxes.

$$m_1 - m_2 = -2.5\log\left(\frac{F_1}{F_2}\right)$$ **1.0**

$$\therefore \frac{F_1}{F_2} = 10^{-0.4(m_1-m_2)} = 10^{-0.4(3.46-4.08)}$$

$$= 1.77$$ **1.0**

$$L_i = 4\pi R_i^2 \sigma T_i^4$$ **1.0**

$$\therefore F_1 = \frac{4\pi R_1^2 \sigma T_1^4}{4\pi D^2} \qquad F_2 = \frac{4\pi R_2^2 \sigma T_2^4}{4\pi D^2}$$ **1.0**

$$\frac{F_1}{F_2} = \frac{R_1^2}{R_2^2} \times \frac{T_1^4}{T_2^4}$$
$$\frac{R_1}{R_2} = \sqrt{\frac{F_1}{F_2}} \times \left(\frac{T_2}{T_1}\right)^2$$

From Wien's displacement law, $\frac{T_2}{T_1} = \frac{\lambda_1}{\lambda_2}$. **1.0**

$$\therefore \frac{R_1}{R_2} = \sqrt{\frac{F_1}{F_2}} \times \left(\frac{\lambda_1}{\lambda_2}\right)^2$$ **1.0**

$$= \sqrt{1.77} \times \left(\frac{531.0}{649.1}\right)^2$$

$$\boxed{\frac{R_1}{R_2} = 0.890}$$ **1.0**

Acceptable range: ± 0.010**.**

(T6.2) Find the radii of the star (in metres) in its most contracted and most expanded states (R_1 and R_2). **3**

Solution:

$$R_2 - R_1 = v \times P/2$$ **2.0**

$$R_2 - R_1 = 12.8 \times 10^3 \times 86\,400 \times \frac{9.84}{2}\,\mathrm{m}$$

$$(1 - 0.890)R_2 = 5.441 \times 10^9\,\mathrm{m}$$

$$\therefore \boxed{R_2 = 4.95 \times 10^{10}\,\mathrm{m}}$$ **0.5**

$$\boxed{R_1 = 4.41 \times 10^{10}\,\mathrm{m}}$$ **0.5**

Acceptable range: $\pm 0.02 \times 10^{10}\,\mathrm{m}$ **for both.**

(T6.3) Calculate the flux of the star, F_2, when it is in its most expanded state. **5**

Solution:

To get the absolute value of flux (F_2) we must compare it with observed flux of the Sun.

$$m_2 - m_\odot = -2.5 \log\left(\frac{F_2}{F_\odot}\right)$$

$$\therefore F_2 = F_\odot 10^{-0.4(m_2 - m_\odot)}$$ **3.0**

$$= \frac{L_\odot}{4\pi a_\oplus^2} \times 10^{-0.4(4.08+26.72)}$$

$$= \frac{3.826 \times 10^{26} \times 4.7863 \times 10^{-13}}{4\pi(1.496 \times 10^{11})^2}\,\mathrm{W\,m^{-2}}$$

$$\boxed{F_2 = 6.51 \times 10^{-10}\,\mathrm{W\,m^{-2}}}$$ **2.0**

Acceptable range: $\pm 0.04 \times 10^{-10}\,\mathrm{W\,m^{-2}}$**.**

(T6.4) Find the distance to the star, $D_{\rm star}$, in parsecs. **5**

Solution:

$$D_{\rm star} = \sqrt{\frac{L_2}{4\pi F_2}} = \sqrt{\frac{R_2^2 \sigma T_2^4}{F_2}} = R_2 T_2^2 \sqrt{\frac{\sigma}{F_2}} \qquad \textbf{2.0}$$

Wien's law: $T_2 = \dfrac{2.898 \times 10^{-3}\,\mathrm{m\,K}}{\lambda_2}$ **1.0**

$$D_{\rm star} = 4.95 \times 10^{10} \times \left(\frac{2.898 \times 10^{-3}}{649.1 \times 10^{-9}}\right)^2 \sqrt{\frac{5.670 \times 10^{-8}}{6.51 \times 10^{-10}}}$$

$\therefore \boxed{D_{\rm star} = 9.208 \times 10^{18}\,\mathrm{m} = 298\,\mathrm{pc}}$ **2.0**

Acceptable range: 298 ± 2pc **(depends on truncation).**

(T7) **Telescope optics**

In a particular ideal refracting telescope of focal ratio $f/5$, the focal length of the objective lens is 100 cm and that of the eyepiece is 1 cm.

(T7.1) What is the angular magnification, m_0, of the telescope? What is the length of the telescope, L_0, i.e. the distance between its objective and eyepiece? **4**

Solution:

The magnification will be given by,

$$m_0 = \frac{f_o}{f_e} \qquad \textbf{1.0}$$

$$= \frac{100}{1} = 100 \qquad \textbf{1.0}$$

The magnification is $\boxed{m_0 = 100}$

Length of the telescope will be

$$L_0 = f_o + f_e \qquad \textbf{1.0}$$

$$= 100 + 1 = 101\,\mathrm{cm} \qquad \textbf{1.0}$$

The telescope length will be $\boxed{L_0 = 101\,\mathrm{cm}}$

Exact answer required for credit.

An introduction of a concave lens (Barlow lens) between the objective lens and the prime focus is a common way to increase the magnification without a large increase in the length of the telescope. A Barlow lens of focal length 1 cm is now introduced between the objective and the eyepiece to double the magnification.

(T7.2) At what distance, $d_{\rm B}$, from the prime focus must the Barlow lens be kept in order to obtain this desired double magnification? **6**

Solution:

We use the following sign convention. Lens is the origin. Direction along the direction

of light is taken as positive. The lens formula is $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$ (f is positive for convex lens and negative for concave lens). Magnification is $m = \frac{v}{u}$. Solutions using other sign conventions are acceptable.

Let v be image distance from the Barlow lens.

$$\frac{1}{f_\mathrm{B}} = \frac{1}{v} - \frac{1}{u}$$ **1.0**

Distance of Barlow lens, d_B, before the prime focus is same as the object distance, u, in this case. **1.0**

$$\frac{1}{f_\mathrm{B}} = \frac{1}{v} - \frac{1}{d_B}$$

Also, $m_\mathrm{B} = 2 = \frac{v}{u} = \frac{v}{d_\mathrm{B}}$ **0.5**

$$\therefore \frac{1}{d_\mathrm{B}} = \frac{2}{v}$$ **1.0**

$$\therefore \frac{1}{-1} = \frac{1}{v} - \frac{2}{v}$$

$$-1 = \frac{-1}{v}$$

$$v = 1\,\mathrm{cm}$$

$$d_\mathrm{B} = \frac{v}{2} = \frac{1\,\mathrm{cm}}{2} = 0.5\,\mathrm{cm}$$ **2.5**

The positive sign for d_B indicates that the Barlow lens was introduced $\boxed{0.5\,\mathrm{cm}}$ before the prime focus.

(T7.3) What is the increase, ΔL, in the length of the telescope? **4**

Solution:

The increase in the length will be,

$$\Delta L = v - d_\mathrm{B}$$ **2.0**

$$= 1.0 - 0.5 = 0.5\,\mathrm{cm}$$ **2.0**

Thus, the length will be increased by $\boxed{\Delta L = 0.5\,\mathrm{cm}}$

A telescope is now constructed with the same objective lens and a CCD detector placed at the prime focus (without any Barlow lens or eyepiece). The size of each pixel of the CCD detector is $10\,\mu\mathrm{m}$.

(T7.4) What will be the distance in pixels between the centroids of the images of the two stars, n_p, on the CCD, if they are $20''$ apart on the sky? **6**

Solution:

Plate scale at prime focus is given by,

$$s = \frac{1}{f_o} = \frac{1\,\mathrm{rad}}{1\,\mathrm{m}} = 0.206\,265\,\mathrm{arcsec}/\mu\mathrm{m}$$ **2.0**

Since each pixel is 10 µm in size,

$$s_{\rm p} = 10 \times 0.206\,\text{arcsec/µm} = 2.06\,\text{arcsec/pixel}$$ **2.0**

Two stars will be separated by,

$$n_{\rm p} = \frac{20''}{2.06''}\ \text{pixels} \simeq \boxed{10\ \text{pixels}}$$ **2.0**

Acceptable range: 9.5 to 10.5 pixels.

(T8) **U-band photometry**

A star has an apparent magnitude $m_U = 15.0$ in the U-band. The U-band filter is ideal, i.e., it has perfect (100%) transmission within the band and is completely opaque (0% transmission) outside the band. The filter is centered at 360 nm, and has a width of 80 nm. It is assumed that the star also has a flat energy spectrum with respect to frequency. The conversion between magnitude, m, in any band and flux density, f, of a star in Jansky ($1\,\text{Jy} = 1 \times 10^{-26}\,\text{W Hz}^{-1}\,\text{m}^{-2}$) is given by

$$f = 3631 \times 10^{-0.4m}\text{Jy}$$

(T8.1) Approximately how many U-band photons, N_0, from this star will be incident normally on a $1\,\text{m}^2$ area at the top of the Earth's atmosphere every second? **8**

Solution:

The U-band is defined as (360 ± 40) nm. Thus, the maximum, minimum and average frequencies of the band will be,

$$\nu_{\rm max} = \frac{c}{\lambda_{\rm max}} = 9.369 \times 10^{14}\,\text{Hz}$$
$$\nu_{\rm min} = 7.495 \times 10^{14}\,\text{Hz}$$
$$\nu_{\rm avg} = 8.432 \times 10^{14}\,\text{Hz}$$
$$\Delta\nu = \nu_{\rm max} - \nu_{\rm min} = 1.874 \times 10^{14}\,\text{Hz}$$ **2.0**

$$f_{\rm st1} = 3631 \times 10^{-0.4\times15} = 3.631\,\text{mJy} = 3.631 \times 10^{-29}\,\text{W Hz}^{-1}\,\text{m}^{-2}$$ **2.0**

$$\text{Now, } N_0 \times h\nu_{\rm avg} = \Delta\nu \times f_{\rm st1} \times A \times \Delta t$$ **2.0**

$$\text{where, } A = 1\,\text{m}^2\ \&\ \Delta t = 1\,\text{s}$$
$$\therefore N_0 = \frac{1.874 \times 10^{14} \times 3.631 \times 10^{-29}}{6.626 \times 10^{-34} \times 8.432 \times 10^{14}} \simeq \boxed{12180}$$ **2.0**

Exact calculation including integration is accepted with full credit (exact answer: 12190).
Accepted range: 12180 ± 200**.**
Using flat spectrum for $\Delta\lambda$ instead of $\Delta\nu$ is considered a major conceptual error, and will incur penalty of 2.0 marks.

This star is being observed in the U-band using a ground based telescope, whose primary mirror has a diameter of 2.0 m. Atmospheric extinction in U-band during the observation is 50%. You may assume that the seeing is diffraction limited. Average surface brightness of night sky in U-band was measured to be 22.0 mag/arcsec^2.

(T8.2) What is the ratio, R, of number of photons received per second from the star to that received from the sky, when measured over a circular aperture of diameter $2''$? **8**

Solution:

Let us call sky flux per square arcsec as Φ and total sky flux for the given aperture as $\phi_{\rm sky}$. Let total star flux be ϕ_{st}.

$$\phi_{\rm sky} = A\Phi = \pi \times (1\,\text{arcsec})^2 \times \Phi = \pi\Phi \quad \textbf{3.0}$$

$$\therefore m_{\rm sky} = 22.0 + 2.5\log_{10}\left(\frac{\Phi}{\phi_{\rm sky}}\right) \quad \textbf{1.0}$$

$$= 22.0 + 2.5\log_{10}\left(\frac{\cancel{\Phi}}{\pi\cancel{\Phi}}\right)$$

$$= 22.0 - 2.5\log_{10}(\pi)$$

$$m_{\rm sky} = 20.76 \text{ mag} \quad \textbf{1.0}$$

As extinction is 50% **1.0**

$$R = \frac{\phi_{\rm st2}}{\phi_{\rm sky}} = \frac{0.5\phi_{\rm st1}}{\phi_{\rm sky}} = 0.5 \times 10^{(20.76-15)/2.5}$$

$$\simeq \boxed{100} \quad \textbf{2.0}$$

Accepted range: ± 5.

A student may also calculate number of photons incident per second per metre square in this case and then compare it with the answer in the first case to get the correct ratio.

(T8.3) In practice, only 20% of U-band photons falling on the primary mirror are detected. How many photons, $N_{\rm t}$, from the star are detected per second? **4**

Solution:

$$N_{\rm t} \times 1\,\text{m}^2 = N_0 \times 0.5 \times 0.2 \times A_{\rm t} \quad \textbf{2.0}$$

$$N_{\rm t} = 12180 \times 0.5 \times 0.2 \times \pi\left(\frac{2.0}{2}\right)^2 = 1233\pi \quad \textbf{1.0}$$

$$N_{\rm t} \simeq \boxed{3813} \quad \textbf{1.0}$$

Accepted range (3813 ± 50)**.**

(T9) **Mars Orbiter Mission**

India's Mars Orbiter Mission (MOM) was launched using the Polar Satellite Launch Vehicle (PSLV) on 5 November 2013. The dry mass of MOM (body + instruments) was 500 kg and it carried fuel of mass 852 kg. It was initially placed in an elliptical orbit around the Earth with perigee at a height of 264 km and apogee at a height of 23 904 km, above the surface of the Earth. After raising the orbit six times, MOM was transferred to a trans-Mars injection orbit (Hohmann orbit).

The first such orbit-raising was performed by firing the engines for a very short time near the perigee. The engines were fired to change the orbit without changing the plane of the orbit and without changing its perigee. This gave a net impulse of $1.73 \times 10^5\,\text{kg m s}^{-1}$ to the satellite. Ignore the change in mass due to burning of fuel.

(T9.1) What is the height of the new apogee, $h_{\rm a}$, above the surface of the Earth, after this **14**

engine burn?

Solution:

Let apogee and perigee distances be r_a and r_p respectively.

$$r_p = R_\oplus + h_p^i = (6371 + 264)\,\text{km} = 6635\,\text{km} \quad \textbf{1.0}$$

$$r_a = R_\oplus + h_a^i = (6371 + 23904)\,\text{km} = 30\,275\,\text{km} \quad \textbf{1.0}$$

Conservation of energy and angular momentum gives

$$E = -\frac{GmM_\oplus}{r_p + r_a}$$

Total energy at perigee

$$\frac{1}{2}m{v_p}^2 - \frac{GmM_\oplus}{r_p} = E = -\frac{GmM_\oplus}{r_p + r_a}$$

$$\frac{1}{2}{v_p}^2 = \frac{GM_\oplus}{r_p}\left(1 - \frac{r_p}{r_p + r_a}\right)$$

$$\therefore v_p = \sqrt{2\frac{GM_\oplus}{r_a + r_p}\frac{r_a}{r_p}} \quad \textbf{2.0}$$

$$= \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24} \times 3.0275 \times 10^7}{6.635 \times 10^6 \times (3.0275 + 6.635 \times 10^6)}}$$

$$= 9.929\,\text{km}\,\text{s}^{-1} \quad \textbf{1.0}$$

As the engine burn is just 41.6 s, we assume that the entire impulse is applied instantaneously at perigee. The impulse is $J = 1.73 \times 10^5\,\text{kg}\,\text{m}\,\text{s}^{-1}$. Note that the total mass of MOM must include the fuel, so we have to use
$m = 500 + 852 = 1352\,\text{kg}$. **1.0**

Change in velocity due to impulse at perigee is

$$\Delta v = \frac{J}{m} = \frac{1.73 \times 10^5}{1352} = 128.0\,\text{m}\,\text{s}^{-1} \quad \textbf{1.0}$$

The new velocity will be given by (we use $'$ symbol to denote quantities after the first orbit-raising maneuvre)

$$v_p' = v_p + \Delta v = 10.06\,\text{km}\,\text{s}^{-1} \quad \textbf{1.0}$$

The perigee remains unchanged. So we get $r_p' = r_p$. **1.0**

Since the satellite is moving faster, the new apogee will be higher.

$$v_p' = \sqrt{2GM_\oplus \frac{r_a'}{r_p'(r_a' + r_p')}}$$

$$\therefore 1 + \frac{r_p'}{r_a'} = \frac{2GM_\oplus}{(v_p')^2 \times r_p'} \quad \textbf{2.0}$$

$$= \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(10.06 \times 10^3)^2 \times 6.635 \times 10^6} = 1.188$$

$$r_a' = \frac{6635}{0.188} = 35\,380\,\text{km} \quad \textbf{2.0}$$

$$h_a = 35380 - 6371$$

$$= \boxed{29\,009\,\text{km}} \quad \textbf{1.0}$$

Acceptable range: $\pm 150\,\text{km}$

(T9.2) Find eccentricity (e) of the new orbit after the burn and new orbital period (P) of MOM in hours. **6**

Solution:

As seen above,

$$\frac{r'_\text{p}}{r'_\text{a}} = 0.188 = \frac{1-e}{1+e} \quad \textbf{1.0}$$

$$\therefore e = \frac{1-0.188}{1+0.188} = \boxed{0.683} \quad \textbf{1.0}$$

Acceptable range: ± 0.002

The new orbital semi-major axis and orbital period will be,

$$a' = \frac{r'_\text{a} + r'_\text{p}}{2} \quad \textbf{1.0}$$

$$= \frac{35380 + 6635}{2} = 20\,933\,\text{km} \quad \textbf{1.0}$$

$$P = 2\pi\sqrt{\frac{a'^3}{GM_\oplus}} \quad \textbf{1.0}$$

$$= 2\pi\sqrt{\frac{(2.0933\times10^7)^3}{6.674\times10^{-11}\times5.972\times10^{24}}} = \boxed{30\,136\,\text{s} = 8.37\,\text{h}} \quad \textbf{1.0}$$

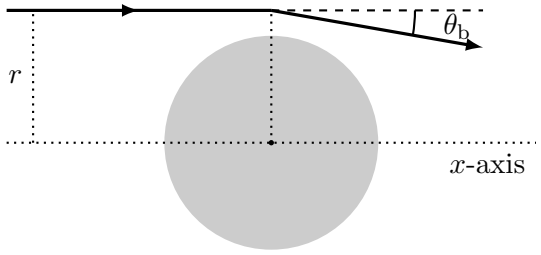
Acceptable range: $\pm 0.1\,\text{h}$

(T10) **Gravitational Lensing Telescope**

Einstein's General Theory of Relativity predicts bending of light around massive bodies. For simplicity, we assume that the bending of light happens at a single point for each light ray, as shown in the figure. The angle of bending, θ_b, is given by

$$\theta_\text{b} = \frac{2R_\text{sch}}{r}$$

where R_sch is the Schwarzschild radius associated with that gravitational body. We call r, the distance of the incoming light ray from the parallel x-axis passing through the centre of the body, as the "impact parameter".



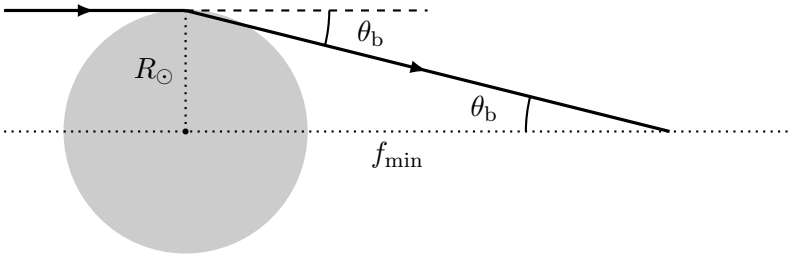
A massive body thus behaves somewhat like a focusing lens. The light rays coming from infinite distance beyond a massive body, and having the same impact parameter r, converge at a point along the axis, at a distance f_r from the centre of the massive body. An observer at that point

will benefit from huge amplification due to this gravitational focusing. The massive body in this case is being used as a Gravitational Lensing Telescope for amplification of distant signals.

(T10.1) Consider the possibility of our Sun as a gravitational lensing telescope. Calculate the shortest distance, $f_{\min}$, from the centre of the Sun (in A.U.) at which the light rays can get focused. **6**

Solution:

The rays travelling closer to the gravitational body will bend more. Thus, we get shortest convergence point where the rays just grazing the solar surface will meet each other.



$$\theta_b = \frac{2R_{sch}}{R_\odot} \simeq \frac{R_\odot}{f_{\min}} \quad \textbf{2.0}$$

$$\therefore f_{\min} = \frac{R_\odot^2}{2R_{sch}} = \frac{R_\odot^2 c^2}{4GM_\odot} \quad \textbf{2.0}$$

$$= \frac{(6.955 \times 10^8 \times 2.998 \times 10^8)^2}{4 \times 6.674 \times 10^{-11} \times 1.989 \times 10^{30}} \text{ m}$$

$$= 8.188 \times 10^{13} \text{ m} = \frac{8.188 \times 10^{13}}{1.496 \times 10^{11}} \text{ AU}$$

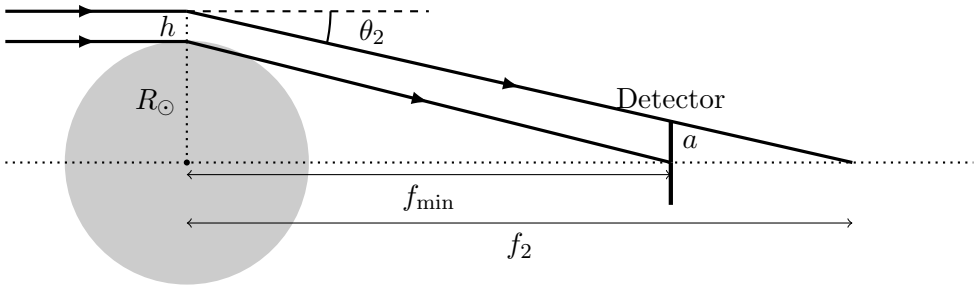
$$\boxed{f_{\min} = 547.3 \text{ AU}} \quad \textbf{2.0}$$

(T10.2) Consider a small circular detector of radius a, kept at a distance $f_{\min}$ centred on the x-axis and perpendicular to it. Note that only the light rays which pass within a certain annulus (ring) of width h (where $h \ll R_\odot$) around the Sun would encounter the detector. The amplification factor at the detector is defined as the ratio of the intensity of the light incident on the detector in the presence of the Sun and the intensity in the absence of the Sun. **8**

Express the amplification factor, A_m, at the detector in terms of $R_\odot$ and a.

Solution:

The following figure needs to be drawn. **1.0**



The light bending from the surface of the Sun ($r = R_\odot$) will intersect the detector at its centre, as it is kept at $f_{\min}$.

The boundary of the detector will be intersected by a light ray with $r = R_\odot + h$. This ray will intersect the x-axis at a distance f_2.

$$f_2 = \frac{(R_\odot + h)^2}{2R_{\rm sch}}$$ **2.0**

Same argument as in Part 1

For small angles,

$$\begin{aligned} a &= (f_2 - f_{\min})\theta_2 \\ &= \left[\frac{(R_\odot + h)^2}{2R_{\rm sch}} - \frac{R_\odot^2}{2R_{\rm sch}}\right]\frac{2R_{\rm sch}}{(R_\odot + h)} = \frac{2R_\odot h + h^2}{R_\odot + h} \\ &\simeq 2h \end{aligned}$$ **2.0**

Let original intensity of the incoming radiation be I_0.

The flux at the detector in the presence of Sun is $\Phi_\odot = I_0\ 2\pi\ R_\odot\ h$ **1.0**

The flux at the detector in the absence of Sun is $\Phi_0 = I_0\ \pi\ a^2$ **1.0**

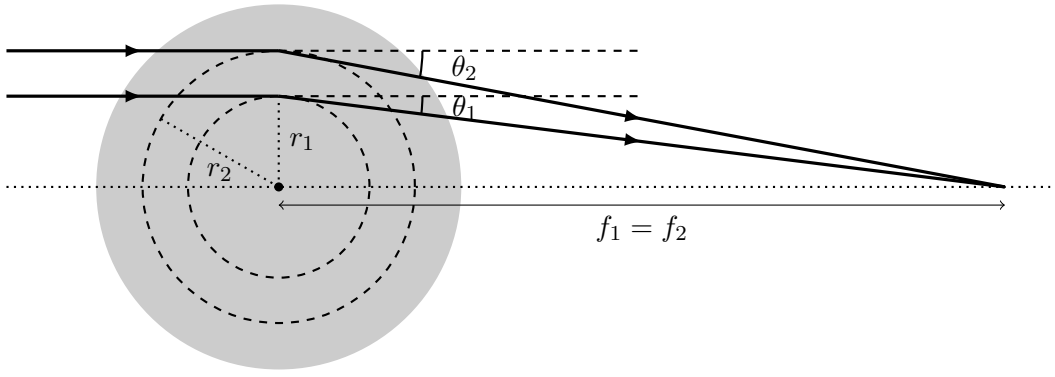
The amplification is therefore

$$A_{\rm m} = \frac{\Phi_\odot}{\Phi_0} = \frac{I_0 2\pi R_\odot h}{I_0 \pi a^2} = \boxed{\frac{R_\odot}{a}}$$ **1.0**

(T10.3) Consider a spherical mass distribution, such as a dark matter cluster, through which light rays can pass while undergoing gravitational bending. Assume for simplicity that for the gravitational bending with impact parameter, r, only the mass $M(r)$ enclosed inside the radius r is relevant. **6**

What should be the mass distribution, $M(r)$, such that the gravitational lens behaves like an ideal optical convex lens ?

Solution:



All rays should focus at the same spot. This should be evident from figure drawn on answersheet or otherwise. **2.0**

Let there be two rays with impact parameters r_1 and r_2. The corresponding distances of focus will be

$$f_i = \frac{r_i^2}{2r_{\rm sch_i}} = \frac{r_i^2 c^2}{4GM(r_i)}$$ **2.0**

Same argument as in Part 1

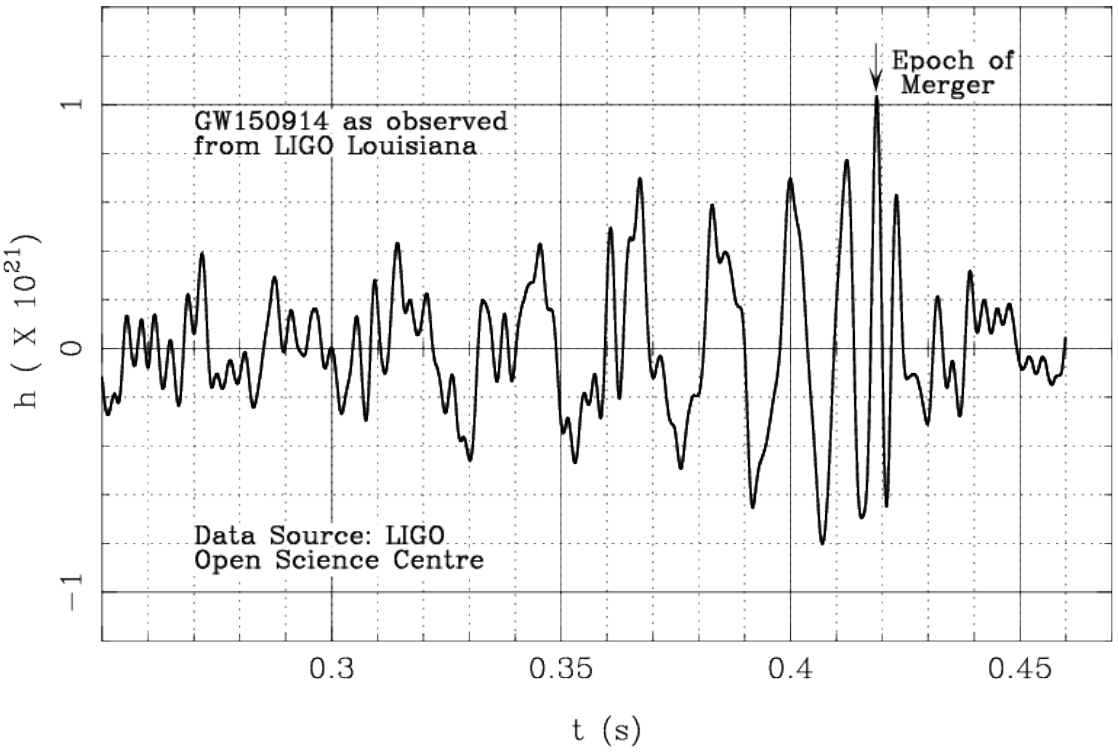
The rquirement $f_1 = f_2$ implies

$$\frac{r_1^2}{r_2^2} = \frac{M(r_1)}{M(r_2)}$$ **1.0**

The required mass distribution is: $\boxed{M(r) \propto r^2}$ **1.0**

(T11) **Gravitational Waves**

The first signal of gravitational waves was observed by two advanced LIGO detectors at Hanford and Livingston, USA in September 2015. One of these measurements (strain vs time in seconds) is shown in the accompanying figure. In this problem, we will interpret this signal in terms of a small test mass m orbiting around a large mass M (i.e., $m \ll M$), by considering several models for the nature of the central mass.



The test mass loses energy due to the emission of gravitational waves. As a result the orbit keeps on shrinking, until the test mass reaches the surface of the object, or in the case of a black hole, the innermost stable circular orbit – ISCO – which is given by $R_{\rm ISCO} = 3R_{\rm sch}$, where $R_{\rm sch}$ is the Schwarzschild radius of the black hole. This is the "epoch of merger". At this point, the amplitude of the gravitational wave is maximum, and so is its frequency, which is always twice the orbital frequency. In this problem, we will only focus on the gravitational waves before the merger, when Kepler's laws are assumed to be valid. After the merger, the form of gravitational waves will drastically change.

(T11.1) Consider the observed gravitational waves shown in the figure above. Estimate the time period, T_0, and hence calculate the frequency, f_0, of gravitational waves just before the epoch of merger. **3**

Solution:

From the graph, just before the peak of emission, the time period of gravitational waves is approximately (0.007 ± 0.004) s.

$T_0 \approx \boxed{0.007\,\text{s}}$ **2.0**

Acceptable range: 0.003 to 0.011 s

That is, the frequency of the gravitational waves is $f_0 \approx \boxed{142.86\,\text{Hz}}$. **1.0**

Acceptable range: 333.33 to 90.91 Hz

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.2) For any main sequence (MS) star, the radius of the star, R_{MS}, and its mass, M_{MS}, are related by a power law given as, **10**

$$R_{\text{MS}} \propto (M_{\text{MS}})^{\alpha}$$
$$\text{where} \quad \alpha = 0.8 \quad \text{for } M_\odot < M_{\text{MS}}$$
$$= 1.0 \quad \text{for } 0.08 M_\odot \leq M_{\text{MS}} \leq M_\odot$$

If the central object were a main sequence star, write an expression for the maximum frequency of gravitational waves, f_{MS}, in terms of mass of the star in units of solar masses $(M_{\text{MS}}/M_\odot)$ and α.

Solution:

Since $m \ll M$ then, by Kepler's third law

$$f_{\text{orbital}} = \frac{1}{2\pi}\sqrt{\frac{GM}{r^3}}$$ **4.0**

Hence the frequency of the gravitational waves is

$$f_{\text{grav}} = 2f_{\text{orbital}} = \frac{1}{\pi}\sqrt{\frac{GM}{r^3}}$$ **1.0**

The frequency will be maximum when $r = R_{\text{MS}}$. **1.0**

For main sequence stars,

$$\frac{R_{\text{MS}}}{R_\odot} = \left(\frac{M_{\text{MS}}}{M_\odot}\right)^{\alpha}$$
$$\therefore R_{\text{MS}} = R_\odot \left(\frac{M_{\text{MS}}}{M_\odot}\right)^{\alpha}$$ **1.0**
$$\therefore f_{\text{MS}} = \frac{1}{\pi}\sqrt{\frac{GM_{\text{MS}}}{R_\odot^3}}\left(\frac{M_\odot}{M_{\text{MS}}}\right)^{3\alpha/2}$$
$$= \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}}\left(\frac{M_\odot}{M_{\text{MS}}}\right)^{(3\alpha-1)/2}$$
$$\boxed{f_{\text{MS}} = \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}}\left(\frac{M_{\text{MS}}}{M_\odot}\right)^{(1-3\alpha)/2}}$$ **3.0**

(T11.3) Using the above result, determine the appropriate value of α that will give the maximum possible frequency of gravitational waves, $f_{\text{MS,max}}$ for any main sequence star. Evaluate **9**

this frequency.

Solution:

For possible values of α given in the question, the exponent $\frac{1-3\alpha}{2}$ is negative. Thus, if $M_{\rm MS} > M_\odot$, the frequency will be smaller. Thus, for highest possible frequency coming from a main sequence star, you should take lowest possible mass i.e. $\boxed{\alpha = 1.0}$ **4.0**

$$f_{\rm MS,max} = \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}}\left(\frac{M_{\rm MS}}{M_\odot}\right)^{\left(\frac{1-3\times1}{2}\right)}$$

$$= \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}} \times \frac{M_\odot}{M_{\rm MS}} \quad \textbf{2.0}$$

The frequency of gravitational waves will be given by,

$$f_{\rm MS,max} = \frac{1}{\pi}\sqrt{\frac{6.674\times10^{-11}\times1.989\times10^{30}}{(6.955\times10^8)^3}} \times \frac{1}{0.08}$$

$\boxed{f_{\rm MS,max} = 2.5\,\text{mHz}}$ **3.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.4) White dwarf (WD) stars have a maximum mass of $1.44M_\odot$ (known as the Chandrasekhar limit) and obey the mass-radius relation $R \propto M^{-1/3}$. The radius of a solar mass white dwarf is equal to 6000 km. Find the highest frequency of emitted gravitational waves, $f_{\rm WD,max}$, if the test mass is orbiting a white dwarf. **8**

Solution:

The maximum frequency would be when $r = R_{\rm WD}$. **1.0**

We use the notation $R_{WD\odot}$ for the radius of a solar mass white dwarf. Then for white dwarfs

$$R_{WD}^3 = R_{WD\odot}^3\frac{M_{WD}}{M_\odot} \quad \textbf{1.0}$$

$$f_{\rm WD} = \frac{1}{\pi}\sqrt{\frac{GM_{\rm WD}}{R_{WD}^3}}$$

$$= \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_{WD\odot}^3}}\frac{M_{WD}}{M_\odot} \quad \textbf{2.0}$$

For maximum frequency, we have to take highest white dwarf mass. **2.0**

$$f_{\rm WD,max} = 2.600\times10^{-6}\times\sqrt{\frac{1.989\times10^{30}}{(6000\times10^3)^3}}\times1.44$$

$$= 2.600\times10^{-6}\times95.96\times10^3\times1.44$$

$\boxed{f_{\rm WD,max} = 0.359\,\text{Hz}}$ **2.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.5) Neutron stars (NS) are a peculiar type of compact objects which have masses between **8**

1 and $3M_\odot$ and radii in the range 10 – 15 km. Find the range of frequencies of emitted gravitational waves, $f_{\rm NS,min}$ and $f_{\rm NS,max}$, if the test mass is orbiting a neutron star at a distance close to the neutron star radius.

Solution:

$$f_{grav} = \frac{1}{\pi}\sqrt{\frac{GM_{\rm NS}}{R_{\rm NS}^3}}$$

The lowest possible frequency is when $M_{\rm NS}$ is lowest and $R_{\rm NS}$ is the highest. **2.0**

For $M_{\rm NS} = M_\odot$ and $R = 15\,$km, we get

$\boxed{f_{\rm NS,min} = 1.996\,\text{kHz}}$ **2.0**

Similarly, the largest possible frequency is when $M_{\rm NS}$ is the largest and $R_{\rm NS}$ is the smallest. **2.0**

For $M_{\rm NS} = 3M_\odot$ and $R = 10\,$km, we get

$\boxed{f_{\rm NS,max} = 6.352\,\text{kHz}}$ **2.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.6) If the test mass is orbiting a black hole (BH), write the expression for the frequency of emitted gravitational waves, $f_{\rm BH}$, in terms of mass of the black hole, $M_{\rm BH}$, and the solar mass $M_\odot$. **7**

Solution:

For black holes, we have to consider R_{ISCO}. **2.0**

Hence the equation will be,

$$f_{BH} = \frac{1}{\pi} \times \sqrt{\frac{GM_\odot}{27R_{sch-\odot}^3}} \times \frac{M_\odot}{M_{BH}}$$ **2.0**

$$\boxed{f_{BH} = 4.396\,\text{kHz} \times \frac{M_\odot}{M_{BH}}}$$ **3.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.7) Based only on the time period (or frequency) of gravitational waves before the epoch of merger, determine whether the central object can be a main sequence star (MS), a white dwarf (WD), a neutron star (NS), or a black hole (BH). Tick the correct option in the Summary Answersheet. Estimate the mass of this object, $M_{\rm obj}$, in units of $M_\odot$. **5**

Solution:

We found the frequency of the LIGO-detected wave to be 166.67 Hz just before merger. As per our analysis above, only black holes can lead to emission in this frequency range.

$\boxed{\text{Black Hole}}$ **2.0**

By using corresponding expression,

$$\boxed{M_{\rm obj} = \frac{4396}{142.86} M_\odot \approx 31 M_\odot}$$ **3.0**

Any answer between 13 to 50 will get full credit.

(T12) **Exoplanets**

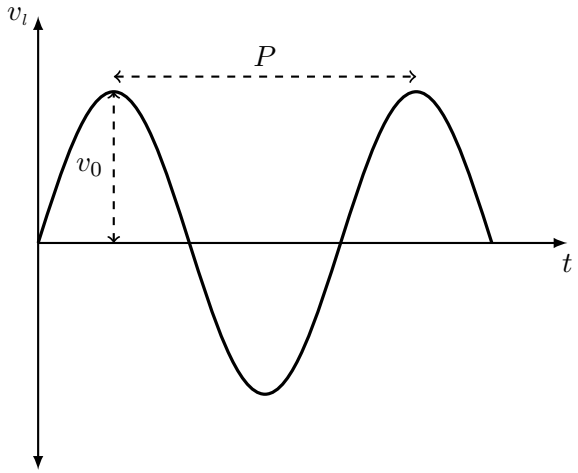
Two major methods of detection of exoplanets (planets around stars other than the Sun) are the radial velocity (or so-called "wobble") method and the transit method. In this problem, we find out how a combination of the results of these two methods can reveal a lot of information about an orbiting exoplanet and its host star.

Throughout this problem, we consider the case of a planet of mass M_p and radius R_p moving in a circular orbit of radius a around a star of mass M_s ($M_\mathrm{s} \gg M_\mathrm{p}$) and radius R_s. The normal to the orbital plane of the planet is inclined at angle i with respect to the line of sight ($i = 90°$ would mean "edge on" orbit). We assume that there is no other planet orbiting the star and $R_\mathrm{s} \ll a$.

"Wobble" Method:

When a planet and a star orbit each other around their barycentre, the star is seen to move slightly, or "wobble", since the centre of mass of the star is not coincident with the barycentre of the star-planet system. As a result, the light received from the star undergoes a small Doppler shift related to the velocity of this wobble.

The line of sight velocity, v_l, of the star can be determined from the Doppler shift of a known spectral line, and its periodic variation with time, t, is shown in the schematic diagram below. In the diagram, the two measurable quantities in this method, namely, the orbital period P and maximum line of sight velocity v_0 are shown.



(T12.1) Derive expressions for the orbital radius (a) and orbital speed (v_p) of the planet in terms of M_s and P. **3**

Solution:

Kepler's law:

$$\boxed{a = \left(\frac{GM_\mathrm{s}}{4\pi^2}P^2\right)^{1/3}} \quad \textbf{1.0}$$

Gravitational force provides centripetal acceleration:

$$v_\mathrm{p} = \sqrt{\frac{GM_\mathrm{s}}{a}} \quad \textbf{0.5}$$

$$\boxed{v_\mathrm{p} = \left(\frac{2\pi GM_\mathrm{s}}{P}\right)^{1/3}} \quad \textbf{1.5}$$

(T12.2) Obtain a lower limit on the mass of the planet, $M_{\mathrm{p,\,min}}$ in terms of M_{s}, v_0 and v_{p}. **4**

Solution:

Momentum conservation:

$$M_{\mathrm{p}} v_{\mathrm{p}} = M_{\mathrm{s}} v_{\mathrm{s}}$$ **1.5**

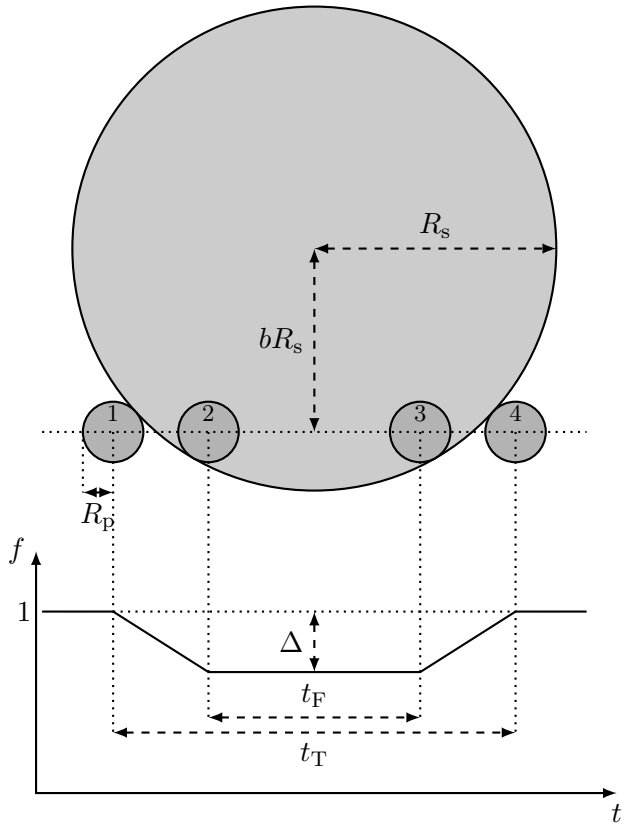
Observed quantity is $v_0 = v_{\mathrm{s}} \sin i$. Thus,

$$M_{\mathrm{p,\,min}} = M_{\mathrm{p}} \sin i = \frac{M_{\mathrm{s}} v_{\mathrm{s}} \sin i}{v_{\mathrm{p}}} = \boxed{\frac{M_{\mathrm{s}} v_0}{v_{\mathrm{p}}}}$$

This is a lower limit on M_{p}. **2.5**

Transit Method:

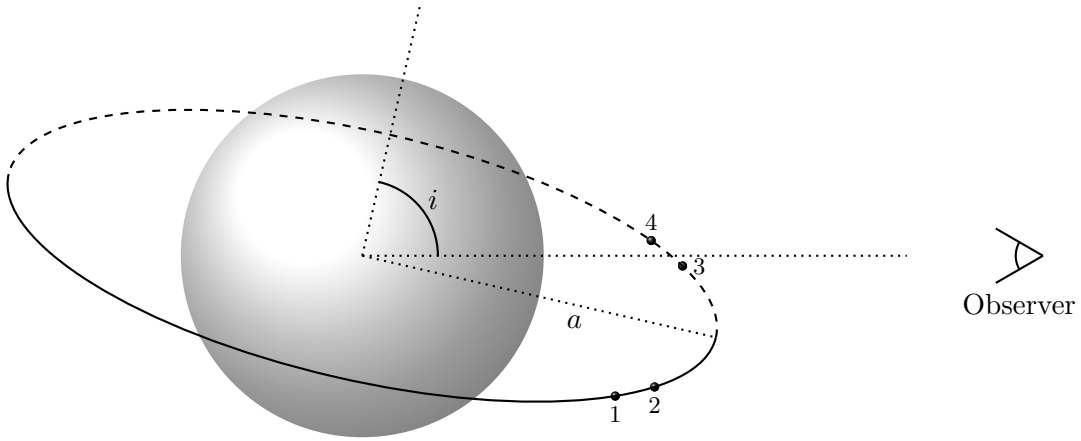
As a planet orbits its host star, for orientations of the orbital plane that are close to "edge-on" ($i \approx 90°$), it will pass periodically, or "transit", in front of the stellar disc as seen by the observer. This would cause a tiny decrease in the observed stellar flux which can be measured. The schematic diagram below (NOT drawn to scale) shows the situation from the observer's perspective and the resulting transit light curve (normalised flux, f, vs time, t) for a uniformly bright stellar disc.



If the inclination angle i is exactly 90°, the planet would be seen to cross the stellar disc along a diameter. For other values of i, the transit occurs along a chord, whose centre lies at a distance bR_{s} from the centre of the stellar disc, as shown. The no-transit flux is normalised to 1 and the maximum dip during the transit is given by Δ.

The four significant points in the transit are the first, second, third and fourth contacts, marked by the positions 1 to 4, respectively, in the figure above. The time interval during the second and third contacts is denoted by $t_{\rm F}$, when the disc of the planet overlaps the stellar disc fully. The time interval between the first and fourth contacts is denoted by $t_{\rm T}$.

These points are also marked in the schematic diagram below showing a "side-on" view of the orbit (NOT drawn to scale).



The measurable quantities in the transit method are P, $t_{\rm T}$, $t_{\rm F}$ and Δ.

(T12.3) Find the constraint on i in terms of $R_{\rm s}$ and a for the transit to be visible at all to the distant observer. **2**

Solution:

$$bR_{\rm s} = a\cos i \qquad \textbf{1.0}$$

Therefore, for visibility, $0 \le b \le 1 \Rightarrow i \ge \cos^{-1}(R_{\rm s}/a)$ **1.0**

(T12.4) Express Δ in terms of $R_{\rm s}$ and $R_{\rm p}$. **1**

Solution:

Blackbody $\Rightarrow$ brightness is proportional to area. Since the observer is far away from the star-planet system, size of silhouette of planet on stellar disc is independent of a.

$$\Delta = \left(\frac{R_{\rm p}}{R_{\rm s}}\right)^2 \qquad \textbf{1.0}$$

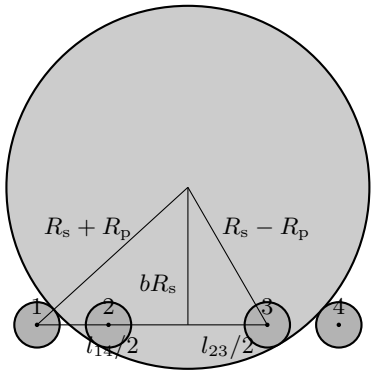
(T12.5) Express $t_{\rm T}$ and $t_{\rm F}$ in terms of $R_{\rm s}$, $R_{\rm p}$, a, P and b. **8**

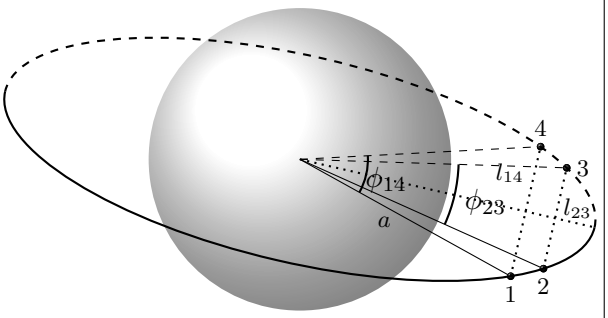
Solution:

Circular orbit $\Rightarrow$ uniform orbital speed

$$\Rightarrow \frac{t}{P} = \frac{a\phi}{2\pi a} = \frac{\phi}{2\pi} \qquad \textbf{1.0}$$

where ϕ is the angle subtended by the planet at the centre of the star during transit (over time t).





$$(l_{23}/2)^2 = (R_s - R_p)^2 - (bR_s)^2$$

$$l_{23} = 2R_s\sqrt{(1 - R_p/R_s)^2 - b^2} \quad \textbf{2.0}$$

$$\sin(\phi_{23}/2) = \frac{l_{23}/2}{a} \quad \textbf{2.0}$$

$$\Rightarrow \phi_{23} = 2\sin^{-1}\left(\frac{l_{23}}{2a}\right) = 2\sin^{-1}\left[\frac{R_s}{a}\sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2}\right] \quad \textbf{1.0}$$

$$t_F = \frac{P}{2\pi}\phi_{23} = \frac{P}{\pi}\sin^{-1}\left[\frac{R_s}{a}\sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2}\right] \quad \textbf{1.0}$$

Similarly,

$$t_T = \frac{P}{\pi}\sin^{-1}\left[\frac{R_s}{a}\sqrt{\left(1 + \frac{R_p}{R_s}\right)^2 - b^2}\right] \quad \textbf{1.0}$$

(T12.6) In the approximation of an orbit much larger than the stellar radius, show that the parameter b is given by **5**

$$b = \left[1 + \Delta - 2\sqrt{\Delta}\frac{1 + \left(\frac{t_F}{t_T}\right)^2}{1 - \left(\frac{t_F}{t_T}\right)^2}\right]^{1/2}$$

Solution:

Since $R_s \ll a$, use $\sin^{-1} x \approx x$. **2.0**

$$t_T \approx \frac{P}{\pi}\left[\frac{R_s}{a}\sqrt{\left(1 + \frac{R_p}{R_s}\right)^2 - b^2}\right]$$

$$t_F \approx \frac{P}{\pi}\left[\frac{R_s}{a}\sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2}\right]$$

Dividing, and putting $R_{\rm p}/R_{\rm s} = \sqrt{\Delta}$,

$$\frac{t_{\rm F}}{t_{\rm T}} = \left[\frac{(1-\sqrt{\Delta})^2 - b^2}{(1+\sqrt{\Delta})^2 - b^2}\right]^{1/2} \quad \textbf{1.0}$$

$$\Rightarrow b = \left[\frac{(1-\sqrt{\Delta})^2 - \left(\frac{t_{\rm F}}{t_{\rm T}}\right)^2 (1+\sqrt{\Delta})^2}{1 - \left(\frac{t_{\rm F}}{t_{\rm T}}\right)^2}\right]^{1/2} = \left[1 + \Delta - 2\sqrt{\Delta}\frac{1 + \left(\frac{t_{\rm F}}{t_{\rm T}}\right)^2}{1 - \left(\frac{t_{\rm F}}{t_{\rm T}}\right)^2}\right]^{1/2} \quad \textbf{2.0}$$

Expressions lacking the use of approximation $R_{\rm s}/a \ll 1$, but otherwise correct will get a penalty of 2.0.
Use of approximation with proper justification at a later stage than at the first step will get full credit.

(T12.7) Use the result of part (T12.6) to obtain an expression for the ratio $a/R_{\rm s}$ in terms of measurable transit parameters, using a suitable approximation. **3**

Solution:

$$t_{\rm T} = \frac{P}{\pi}\left[\frac{R_{\rm s}}{a}\sqrt{\left(1 + \frac{R_{\rm p}}{R_{\rm s}}\right)^2 - b^2}\right] \quad \textbf{1.0}$$

Either substitution of b or elimination of b gets 1.0.
Substituting b and $R_{\rm p}/R_{\rm s}$,

$$t_{\rm T} = \frac{P}{\pi}\left[\frac{R_{\rm s}}{a}\sqrt{(1+\sqrt{\Delta})^2 - 1 - \Delta + 2\sqrt{\Delta}\frac{1 + \left(\frac{t_{\rm F}}{t_{\rm T}}\right)^2}{1 - \left(\frac{t_{\rm F}}{t_{\rm T}}\right)^2}}\right]$$

$$\Rightarrow t_{\rm T} = \frac{P}{\pi}\frac{R_{\rm s}}{a}\left[\frac{4\sqrt{\Delta}}{1 - \left(\frac{t_{\rm F}}{t_{\rm T}}\right)^2}\right]^{1/2}$$

$$\Rightarrow \frac{a}{R_{\rm s}} = \frac{2P\Delta^{1/4}}{\pi(t_{\rm T}^2 - t_{\rm F}^2)^{1/2}} \quad \textbf{2.0}$$

Expressions lacking the use of approximation $R_{\rm s}/a \ll 1$, but otherwise correct will get a penalty of 1.0.
If penalty has already been imposed in part (T12.6), no further penalty for lack of approximation.

(T12.8) Combine the results of the wobble method and the transit method to determine the stellar mean density $\rho_{\rm s} \equiv \dfrac{M_{\rm s}}{4\pi R_{\rm s}^3/3}$ in terms of $t_{\rm T}$, $t_{\rm F}$, Δ and P. **6**

Solution:

From part (T12.7)

$$a = R_{\rm s}\frac{2P\Delta^{1/4}}{\pi(t_{\rm T}^2 - t_{\rm F}^2)^{1/2}}$$

From part (T12.1)

$$a = \left(\frac{GM_{\rm s}}{4\pi^2}P^2\right)^{1/3}$$

Combining,

$$\frac{GM_{\rm s}}{4\pi^2}P^2 = \left[R_{\rm s}\frac{2P\Delta^{1/4}}{\pi(t_{\rm T}^2 - t_{\rm F}^2)^{1/2}}\right]^3$$ **3.0**

Identifying the two equations for combining gets credit. No credit for writing only one equation or irrelevant equations.

$$\Rightarrow \rho_{\rm s} \equiv \frac{M_{\rm s}}{4\pi R_{\rm s}^3/3} = \frac{3}{4\pi}\frac{4\pi^2}{P^2G}\frac{8P^3\Delta^{3/4}}{\pi^3(t_{\rm T}^2 - t_{\rm F}^2)^{3/2}}$$ **2.0**

$$\Rightarrow \boxed{\rho_{\rm s} = \frac{24}{\pi^2 G}\frac{P(\Delta)^{3/4}}{(t_{\rm T}^2 - t_{\rm F}^2)^{3/2}}}$$ **1.0**

Rocky or gaseous:

Let us consider an edge-on ($i = 90°$) star-planet system (circular orbit for the planet), as seen from the Earth. It is known that the host star is of mass $1.00M_\odot$. Transits are observed with a period (P) of 50.0 days and total transit duration ($t_{\rm T}$) of 1.00 hour. The transit depth (Δ) is 0.0064. The same system is also observed in the wobble method to have a maximum line of sight velocity of $0.400\,{\rm m\,s^{-1}}$.

(T12.9) Find the orbital radius a of the planet in units of AU and in metres. **2**

Solution:

From Kepler's third law (with same mass of host star):

$$\frac{a}{a_\oplus} = \left(\frac{P}{P_\oplus}\right)^{2/3}$$ **1.0**

$$a = \left(\frac{50.0}{365.242}\right)^{2/3} \times 1\,{\rm AU} = \boxed{0.266\,{\rm AU}}$$ **0.5**

$$= 0.266 \times 1.496 \times 10^{11}\,{\rm m} = \boxed{3.97 \times 10^{10}\,{\rm m}}$$ **0.5**

(T12.10) Find the ratio $t_{\rm F}/t_{\rm T}$ of the system. **2**

Solution:

Edge-on $\Rightarrow b = 0$ **1.0**

$$\frac{t_{\rm F}}{t_{\rm T}} = \left[\frac{(1-\sqrt{\Delta})^2 - b^2}{(1+\sqrt{\Delta})^2 - b^2}\right]^{1/2} = \frac{1-\sqrt{\Delta}}{1+\sqrt{\Delta}} = \boxed{0.8519}$$ **1.0**

(T12.11) Obtain the mass $M_{\rm p}$ and radius $R_{\rm p}$ of the planet in terms of the mass ($M_\oplus$) and radius ($R_\oplus$) of the Earth respectively. Is the composition of the planet likely to be rocky or gaseous? Tick the box for ROCKY or GASEOUS in the Summary Answersheet. **8**

Solution:

From parts (T12.1) and (T12.9)

$$v_{\rm p} = \sqrt{\frac{GM_\odot}{a}} = \sqrt{\frac{6.674\times10^{-11}\times1.989\times10^{30}}{3.97\times10^{10}}} = 57.798\,{\rm km\,s^{-1}}$$ **1.0**

Assumption of small planet ($M_{\rm p} \ll M_{\rm s}$) is valid because Δ is very small; less dense planet would make the assumption stronger!

$$M_{\rm p} = \frac{M_{\rm s}v_0}{v_{\rm p}} = \frac{1.989\times10^{30}\times0.400}{5.7798\times10^4\times5.972\times10^{24}}M_\oplus = \boxed{2.30\,M_\oplus}$$ **2.0**

From part (T12.4),

$$R_{\rm p} = R_{\rm s}\sqrt{\Delta}$$

From part (T12.7),

$$\frac{a}{R_{\rm s}} = \frac{2P\Delta^{1/4}}{\pi(t_{\rm T}^2 - t_{\rm F}^2)^{1/2}} = \frac{2P\Delta^{1/4}}{\pi t_{\rm T}(1-(t_{\rm F}/t_{\rm T})^2)^{1/2}}$$

$$\therefore R_{\rm s} = \frac{a\pi t_{\rm T}(1-(t_{\rm F}/t_{\rm T})^2)^{1/2}}{2P\Delta^{1/4}}$$

Combining,

$$\begin{aligned} R_{\rm p} &= \frac{a\pi t_{\rm T}(1-(t_{\rm F}/t_{\rm T})^2)^{1/2}\Delta^{1/2}}{2P\Delta^{1/4}} \\ &= \frac{a\pi t_{\rm T}(1-(t_{\rm F}/t_{\rm T})^2)^{1/2}\Delta^{1/4}}{2P} \\ &= \frac{3.97\times10^{10}\times\pi\times\frac{1}{24}\times(1-0.8519^2)^{1/2}\times(0.0064)^{1/4}}{2\times50.0\times6.371\times10^6}R_\oplus \\ &= \boxed{1.21\,R_\oplus} \end{aligned}$$ **2.0** **1.0**

Mean density

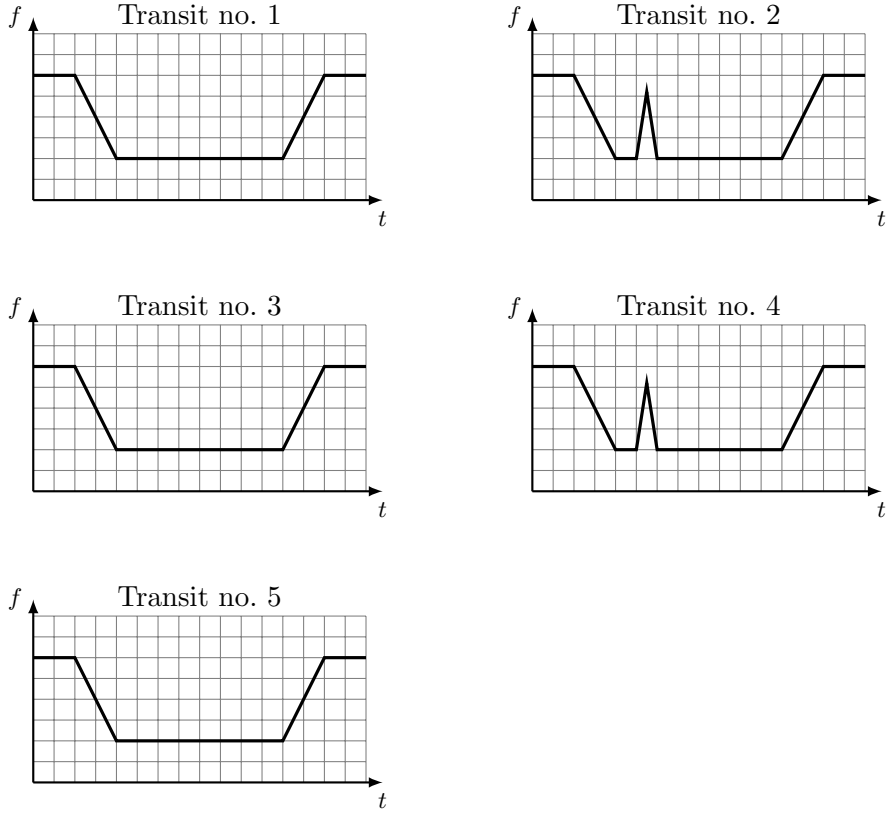
$$\rho_{\rm p} = \frac{M_{\rm p}}{4\pi R_{\rm p}^3/3} = \frac{2.30}{(1.21)^3}\rho_\oplus = 1.3\rho_\oplus$$ **1.0**

Since mean density is higher than that of Earth, the planet is $\boxed{\text{Rocky}}$. **1.0**

Transit light curves with starspots and limb darkening:

(T12.12) Consider a planetary transit with $i = 90^\circ$ around a star which has a starspot on its equator, comparable to the size of the planet, $R_{\rm p}$. The rotation period of the star is $2P$. Draw schematic diagrams of the transit light curve for five successive transits of the planet (in the templates provided in the Summary Answersheet). The no-transit flux for each transit may be normalised to unity independently. Assume that the planet does not encounter the starspot on the first transit but does in the second. **4**

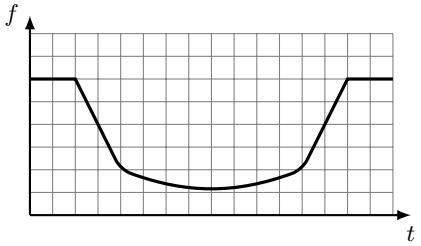
Solution:



- **No change in the first: 0.5**
- **Spike in second (width and phase of spike is arbitrary): 1.0**
- **Height of spike (almost) equal to maximum dip: 0.5**
- **No change in third: 0.5**
- **Spike again in fourth: 0.5**
- **Same phase of spike in second and fourth: 0.5**
- **No change in 5th: 0.5**

(T12.13) Throughout the problem we have considered a uniformly bright stellar disc. However, real stellar discs have limb darkening. Draw a schematic transit light curve when limb darkening is present in the host star. **2**

Solution:



Non-flat bottom with central minimum gets 2.0. Curvature of ingress and egress are tolerated.