Theoretical Examination

(T1) True or False

Determine if each of the following statements is True or False. In the Summary Answersheet, tick the correct answer (TRUE / FALSE) for each statement. No justifications are necessary for this question.

(T1.1) In a photograph of the clear sky on a Full Moon night with a sufficiently long exposure, the colour of the sky would appear blue as in daytime. **2**

(T1.2) An astronomer at Bhubaneswar marks the position of the Sun on the sky at 05: 00 UT every day of the year. If the Earth's axis were perpendicular to its orbital plane, these positions would trace an arc of a great circle. **2**

(T1.3) If the orbital period of a certain minor body around the Sun in the ecliptic plane is less than the orbital period of Uranus, then its orbit must necessarily be fully inside the orbit of Uranus. **2**

(T1.4) The centre of mass of the solar system is inside the Sun at all times. **2**

(T1.5) A photon is moving in free space. As the Universe expands, its momentum decreases. **2**

(T2) Gases on Titan **10**

Gas particles in a planetary atmosphere have a wide distribution of speeds. If the r.m.s. (root mean square) thermal speed of particles of a particular gas exceeds $1/6$ of the escape speed, then most of that gas will escape from the planet. What is the minimum atomic weight (relative atomic mass), A_{min}, of an ideal monatomic gas so that it remains in the atmosphere of Titan?

Given, mass of Titan $M_{\mathrm{T}} = 1.23 \times 10^{23}$ kg, radius of Titan $R_{\mathrm{T}} = 2575$ km, surface temperature of Titan $T_{\mathrm{T}} = 93.7$ K.

(T3) Early Universe

Cosmological models indicate that radiation energy density, ρ_{r}, in the Universe is proportional to $(1+z)^4$, and the matter energy density, ρ_{m}, is proportional to $(1+z)^3$, where z is the redshift. The dimensionless density parameter, Ω, is given as $\Omega = \rho/\rho_{\mathrm{c}}$, where ρ_{c} is the critical energy density of the Universe. In the present Universe, the density parameters corresponding to radiation and matter, are $\Omega_{\mathrm{r_0}} = 10^{-4}$ and $\Omega_{\mathrm{m_0}} = 0.3$, respectively.

(T3.1) Calculate the redshift, z_{e}, at which radiation and matter energy densities were equal. **3**

(T3.2) Assuming that the radiation from the early Universe has a blackbody spectrum with a temperature of 2.732 K, estimate the temperature, T_{e}, of the radiation at redshift z_{e}. **4**

(T3.3) Estimate the typical photon energy, E_{ν} (in eV), of the radiation as emitted at redshift z_{e}. **3**

(T4) Shadows **10**

An observer in the northern hemisphere noticed that the length of the shortest shadow of a 1.000 m vertical stick on a day was 1.732 m. On the same day, the length of the longest shadow of the same vertical stick was measured to be 5.671 m.

Find the latitude, ϕ, of the observer and declination of the Sun, $\delta_{\odot}$, on that day. Assume the Sun to be a point source and ignore atmospheric refraction.

(T5) GMRT beam transit **10**

Giant Metrewave Radio Telescope (GMRT), one of the world's largest radio telescopes at metre wavelengths, is located in western India (latitude: $19^{\circ}6'$ N, longitude: $74^{\circ}3'$ E). GMRT consists of 30 dish antennas, each with a diameter of 45.0 m. A single dish of GMRT was held fixed with its axis pointing at a zenith angle of $39^{\circ}42'$ along the northern meridian such that a radio point source would pass along a diameter of the beam, when it is transiting the meridian.

What is the duration T_{transit} for which this source would be within the FWHM (full width at half maximum) of the beam of a single GMRT dish observing at 200 MHz?

Hint: The FWHM size of the beam of a radio dish operating at a given frequency corresponds to the angular resolution of the dish. Assume uniform illumination.

(T6) Cepheid Pulsations

The star β-Doradus is a Cepheid variable star with a pulsation period of 9.84 days. We make a simplifying assumption that the star is brightest when it is most contracted (radius being R_1) and it is faintest when it is most expanded (radius being R_2). For simplicity, assume that the star maintains its spherical shape and behaves as a perfect black body at every instant during the entire cycle. The bolometric magnitude of the star varies from 3.46 to 4.08. From Doppler measurements, we know that during pulsation the stellar surface expands or contracts at an average radial speed of 12.8 km s^{-1}. Over the period of pulsation, the peak of thermal radiation (intrinsic) of the star varies from 531.0 nm to 649.1 nm.

(T6.1) Find the ratio of radii of the star in its most contracted and most expanded states (R_1/R_2). **7**

(T6.2) Find the radii of the star (in metres) in its most contracted and most expanded states (R_1 and R_2). **3**

(T6.3) Calculate the flux of the star, F_2, when it is in its most expanded state. **5**

(T6.4) Find the distance to the star, D_{star}, in parsecs. **5**

(T7) Telescope optics

In a particular ideal refracting telescope of focal ratio $f/5$, the focal length of the objective lens is 100 cm and that of the eyepiece is 1 cm.

(T7.1) What is the angular magnification, m_0, of the telescope? What is the length of the telescope, L_0, i.e. the distance between its objective and eyepiece? **4**

An introduction of a concave lens (Barlow lens) between the objective lens and the prime focus is a common way to increase the magnification without a large increase in the length of the telescope. A Barlow lens of focal length 1 cm is now introduced between the objective and the eyepiece to double the magnification.

(T7.2) At what distance, d_{B}, from the prime focus must the Barlow lens be kept in order to obtain this desired double magnification? **6**

(T7.3) What is the increase, ΔL, in the length of the telescope? **4**

A telescope is now constructed with the same objective lens and a CCD detector placed at the prime focus (without any Barlow lens or eyepiece). The size of each pixel of the CCD detector is 10 μm.

(T7.4) What will be the distance in pixels between the centroids of the images of the two stars , n_{p}, on the CCD, if they are $20''$ apart on the sky? **6**

(T8) U-Band photometry

A star has an apparent magnitude $m_{\text{U}} = 15.0$ in the *U*-band. The *U*-band filter is ideal, i.e., it has perfect (100%) transmission within the band and is completely opaque (0% transmission) outside the band. The filter is centered at 360 nm, and has a width of 80 nm. It is assumed that the star also has a flat energy spectrum with respect to frequency. The conversion between magnitude, m, in any band and flux density, f, of a star in Jansky ($1\ \text{Jy} = 1 \times 10^{-26}$ W Hz^{-1} m^{-2}) is given by

$$f = 3631 \times 10^{-0.4m}\text{Jy}$$

(T8.1) Approximately how many *U*-band photons, N_0, from this star will be incident normally on a $1\ \text{m}^2$ area at the top of the Earth's atmosphere every second? **8**

This star is being observed in the *U*-band using a ground based telescope, whose primary mirror has a diameter of 2.0 m. Atmospheric extinction in *U*-band during the observation is 50%. You may assume that the seeing is diffraction limited. Average surface brightness of night sky in *U*-band was measured to be $22.0\ \text{mag/arcsec}^2$.

(T8.2) What is the ratio, R, of number of photons received per second from the star to that received from the sky, when measured over a circular aperture of diameter $2''$? **8**

(T8.3) In practice, only 20% of U-band photons falling on the primary mirror are detected. How many photons, N_{t}, from the star are detected per second? **4**

(T9) **Mars Orbiter Mission**

India's Mars Orbiter Mission (MOM) was launched using the Polar Satellite Launch Vehicle (PSLV) on 5 November 2013. The dry mass of MOM (body + instruments) was 500 kg and it carried fuel of mass 852 kg. It was initially placed in an elliptical orbit around the Earth with perigee at a height of 264.1 km and apogee at a height of 23903.6 km, above the surface of the Earth. After raising the orbit six times, MOM was transferred to a trans-Mars injection orbit (Hohmann orbit).

The first such orbit-raising was performed by firing the engines for a very short time near the perigee. The engines were fired to change the orbit without changing the plane of the orbit and without changing its perigee. This gave a net impulse of 1.73×10^5 kg m s^{-1} to the satellite. Ignore the change in mass due to burning of fuel.

(T9.1) What is the height of the new apogee, h_{a} above the surface of the Earth, after this engine burn? **14**

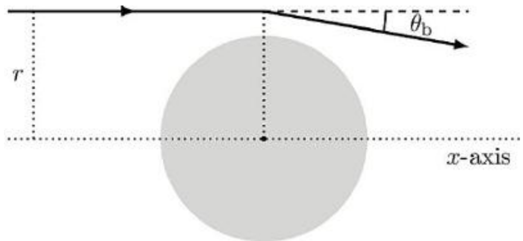
(T9.2) Find the eccentricity (e) of the new orbit after the burn and the new orbital period (P) of MOM in hours. **6**

(T10) **Gravitational Lensing Telescope**

Einstein's General Theory of Relativity predicts bending of light around massive bodies. For simplicity, we assume that the bending of light happens at a single point for each light ray, as shown in the figure. The angle of bending, θ_{b}, is given by

$$\theta_{\mathrm{b}} = \frac{2R_{\mathrm{sch}}}{r}$$

where R_{sch} is the Schwarzschild radius associated with that gravitational body. We call r, the distance of the incoming light ray from the parallel x-axis passing through the centre of the body, as the "impact parameter".



A massive body thus behaves somewhat like a focusing lens. The light rays coming from infinite distance beyond a massive body, and having the same impact parameter r, converge at a point along the axis, at a distance f_r from the centre of the massive body. An observer at that point will benefit from huge amplification due to this gravitational focusing. The massive body in this case is being used as a Gravitational Lensing Telescope for amplification of distant signals.

(T10.1) Consider the possibility of our Sun as a gravitational lensing telescope. Calculate the shortest distance, f_{min}, from the centre of the Sun (in A. U.) at which the light rays can get focused. **6**

(T10.2) Consider a small circular detector of radius a, kept at a distance f_{min} centered on the x-axis and perpendicular to it. Note that only the light rays which pass within a certain annulus (ring) of width h (where $h \ll R_{\odot}$) around the Sun would encounter the detector. The amplification factor at the detector is defined as the ratio of the intensity of the light incident on the detector in the presence of the Sun and the intensity in the absence of the Sun. **8**

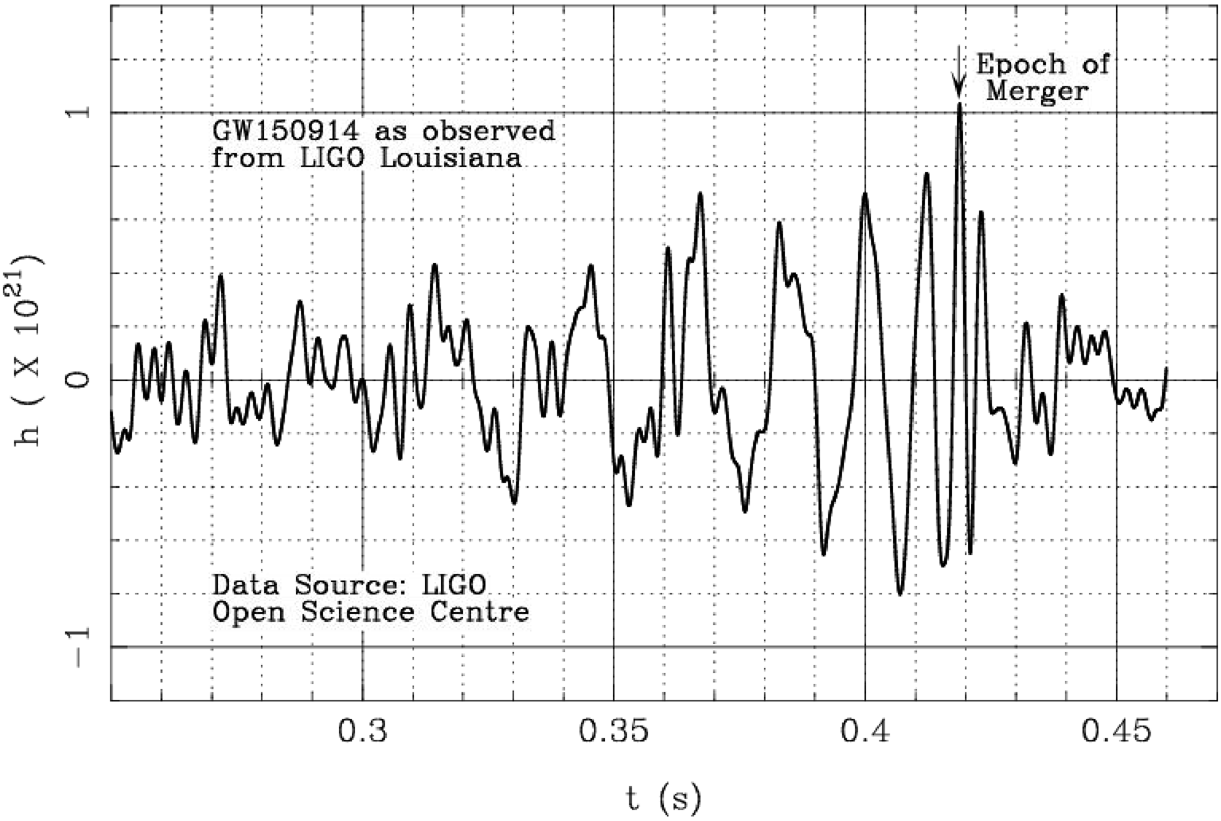
Express the amplification factor, A_{m}, at the detector in terms of $R_{\odot}$ and a.

(T10.3) Consider a spherical mass distribution, such as dark matter in a galaxy cluster, through which light rays can pass while undergoing gravitational bending. Assume for simplicity that for the gravitational bending with impact parameter, r, only the mass $M(r)$ enclosed inside the radius r is relevant. **6**

What should be the mass distribution, $M(r)$, such that the gravitational lens behaves like an ideal optical convex lens?

(T11) **Gravitational Waves**

The first signal of gravitational waves was observed by two advanced LIGO detectors at Hanford and Livingston, USA in September 2015. One of these measurements (strain vs time in seconds) is shown in the accompanying figure. In this problem, we will interpret this signal in terms of a small test mass m orbiting around a large mass M (i.e., $m \ll M$), by considering several models for the nature of the central mass.



The test mass loses energy due to the emission of gravitational waves. As a result the orbit keeps on shrinking, until the test mass reaches the surface of the object, or in the case of a black hole, the innermost stable circular orbit – ISCO – which is given by $R_{\text{ISCO}} = 3R_{\text{sch}}$, where R_{sch} is the Schwarzschild radius of the black hole. This is the "epoch of merger". At this point, the amplitude of the gravitational wave is maximum, and so is its frequency, which is always twice the orbital frequency. In this problem, we will only focus on the gravitational waves before the merger, when Kepler's laws are assumed to be valid. After the merger, the form of gravitational waves will drastically change.

(T11.1) Consider the observed gravitational waves shown in the figure above. Estimate the time period, T_0, and hence calculate the frequency, f_0, of gravitational waves just before the epoch of merger. **3**

(T11.2) For any main sequence (MS) star, the radius of the star, R_{MS}, and its mass, M_{MS}, are related by a power law given as, **10**

$$\begin{aligned} R_{\text{MS}} &\propto (M_{\text{MS}})^{\alpha} \\ \text{where } \alpha &= 0.8 \qquad \text{for } M_{\odot} < M_{\text{MS}} \\ &= 1.0 \qquad \text{for } 0.08M_{\odot} \le M_{\text{MS}} \le M_{\odot} \end{aligned}$$

If the central object were a main sequence star, write an expression for the maximum frequency of gravitational waves, f_{MS}, in terms of mass of the star in units of solar masses ($M_{\text{MS}}/M_{\odot}$) and α.

(T11.3) Using the above result, determine the appropriate value of α that will give the maximum possible frequency of gravitational waves, $f_{\text{MS,max}}$ for any main sequence star. Evaluate this frequency. **9**

(T11.4) White dwarf (WD) stars have a maximum mass of 1.44 $M_{\odot}$ (known as the Chandrasekhar limit) and obey the mass-radius relation $R \propto M^{-1/3}$. The radius of a solar mass white dwarf is equal **8** to 6000 km. Find the highest frequency of emitted gravitational waves, $f_{\mathrm{WD,max}}$, if the test mass is orbiting a white dwarf.

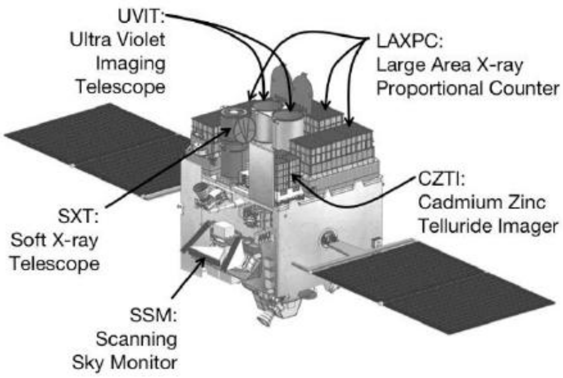
(T11.5) Neutron stars (NS) are a peculiar type of compact objects which have masses between 1 and $3M_{\odot}$ and radii in the range $10 - 15$ km. Find the range of frequencies of emitted gravitational waves, $f_{\mathrm{NS,min}}$ and $f_{\mathrm{NS,max}}$, if the test mass is orbiting a neutron star at a distance close to the neutron star radius. **8**

(T11.6) If the test mass is orbiting a black hole (BH), write the expression for the frequency of emitted gravitational waves, f_{BH}, in terms of mass of the black hole, M_{BH}, and the solar mass $M_{\odot}$. **7**

(T11.7) Based only on the time period (or frequency) of gravitational waves before the epoch of merger, determine whether the central object can be a main sequence star (MS), a white dwarf (WD), a neutron star (NS), or a black hole (BH). Tick the correct option in the Summary Answersheet. Estimate the mass of this object, M_{obj}, in units of $M_{\odot}$. **5**

(T12) **AstroSat**

India astronomy satellite, AstroSat, launched in September 2015, has five different instruments.



In this question, we will discuss three of these instruments (SXT, LAXPC, CZTI), which point in the same direction and observe in X-ray wavelengths. The details of these instruments are given in the table below.

Instrument	Band [keV]	Collecting Area [m^2]	Effective Photon Detection Efficiency	Saturation level [counts]	No. of Pixels
SXT	0.3 – 80	0.067	60%	15000 (total)	512 x 512
LAXPC	3 – 80	1.5	40%	50000 (in any one counter) or 200000 (total)	---
CZTI	10 – 150	0.09	50%	---	4 x 4096

You should note that LAXPC energy range is divided into 8 different energy band counters of equal bandwidth with no overlap.

(T12.1) Some X-ray sources like Cas A have a prominent emission line at 0.01825 nm corresponding to a radioactive transition of ^{44}Ti. Suppose there exists a source which emits only one bright emission line corresponding to this transition. What should be the minimum relative velocity (v) of the source, which will make the observed peak of this line to get registered in a different energy band counter of LAXPC as compared to a source at rest? **13**

These instruments were used to observe an X-ray source (assumed to be a point source), whose energy spectrum followed the power law,

$$F(E) = KE^{-2/3} \qquad \text{[in units of counts/keV/m}^2\text{/s]}$$

where E is the energy in keV, K is a constant and $F(E)$ is photon flux density at that energy. Photon flux density, by definition, is given for per unit collecting area (m^2) per unit bandwidth (keV) and per unit time (seconds). From prior observations, we know that the source has a flux density of 10 $\mathrm{counts/keV/m^2/s}$ at 1 keV, when measured by a detector with 100% photon detection efficiency. The "counts" here mean the number of photons reported by the detector.

As the source flux follows the power law given above, we know that for a given energy range from E_1 (lower energy) to E_2 (higher energy) the total photon flux (F_T) will be given by

$$F_\mathrm{T} = 3K\,(E_2^{1/3} - E_1^{1/3}) \qquad \text{[in units of } \mathrm{counts/m^2/s}\text{]}$$

(T12.2) Estimate the incident flux density from the source at 1 keV, 5 keV, 40 keV and 100 keV. Also estimate what will be the total count per unit bandwidth recorded by each of the instruments at these energies for an exposure time of 200 seconds. **[8]**

(T12.3) For this source, calculate the maximum exposure time (t_S), without suffering from saturation, for the CCD of SXT. **[4]**

(T12.4) If the source became 3500 times brighter, calculate the expected counts per second in LAXPC counter 1, counter 8 as well as total counts across the entire energy range. If we observe for longer period, will the counter saturate due to any individual counter or due to the total count? Tick the appropriate box in the Summary Answersheet. **[8]**

(T12.5) Assume that the counts reported by CZTI due to random fluctuations in electronics are about 0.00014 counts per pixel per keV per second at all energy levels. Any source is considered as "detected" when the SNR (signal to noise ratio) is at least 3. What is minimum exposure time, t_c, needed for the source above to be detected in CZTI? **[10]**
Note that the "noise" in a detector is equal to the square root of the counts due to random fluctuations.

(T12.6) Let us consider the situation where the source shows variability in number flux, so that the factor K increases by 20%. AstroSat observed this source for 1 second before the change and 1 second after this change in brightness. Calculate the counts measured by SXT, LAXPC and CZTI in both the observations. Which instrument is best suited to detect this change? Tick the appropriate box in the Summary Answersheet. **[7]**