LONG QUESTIONS: MARKING SCHEME

1. A moon is orbiting a planet such that the plane of its orbit is perpendicular to the surface of the planet where an observer is standing. After some necessary scaling, suppose the orbit satisfies the following equation:

$$9\left(\frac{x}{2}+\frac{\sqrt{3}y}{2}-4\right)^2+25\left(-\frac{\sqrt{3}x}{2}+\frac{y}{2}\right)^2=225$$

Consider Cartesian coordinates where x is on the horizontal plane and y is on the zenith of the observer. Let r be the radius of the moon. Assume that the period of rotation of the planet is much larger than the orbital period of the moon. Ignore the atmospheric refraction.

a. Calculate the semimajor and semiminor axis of the ellipse.
b. Calculate the zenith angle of perigee.
c. Determine $\tan\frac{\theta}{2}$ where θ is the elevation angle (altitude of the upper tangent of the moon) when the moon looks largest to the observer.

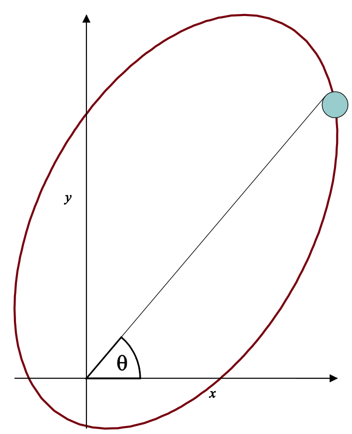
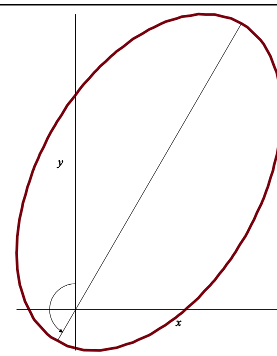


Figure 1

Answer and Marking Scheme:

1. Polar version

(a)	**Rewrite the orbit equation using polar coordinate** $$x = R(\alpha)\cos\alpha$$ $$y = R(\alpha)\sin\alpha$$ Thus, if we let $$x^* = R(\alpha)\cos\left(\alpha - \frac{\pi}{3}\right) = \frac{x}{2} + \frac{y\sqrt{3}}{2}$$ $$y^* = R(\alpha)\sin\left(\alpha - \frac{\pi}{3}\right) = -\frac{x\sqrt{3}}{2} + \frac{y}{2}$$ the equation of the orbit can be written as $$9\left(R(\alpha)\cos\left(\alpha - \frac{\pi}{3}\right) - 4\right)^2 + 25\left(R(\alpha)\sin\left(\alpha - \frac{\pi}{3}\right)\right)^2 = 225$$ $$9(x^* - 4)^2 + 25(y^*)^2 = 225$$	**10** **10**
	Determine semi major and semi minor axis of the orbit Since $$9(x^* - 4)^2 + 25(y^*)^2 = 225$$ is equivalent to $$\frac{(x^* - 4)^2}{25} + \frac{(y^*)^2}{9} = 1$$ then, the semi major axis is 5, and the semi minor axis is 3	**5** **5**
(b)	**Determine the zenith angle at perigee** At the perigee, $\alpha - \frac{\pi}{3} = \pi$. Thus, the zenith angle at the perigee is $$\alpha - \frac{\pi}{2} = \pi + \frac{\pi}{3} - \frac{\pi}{2} = \frac{5\pi}{6}$$.  	**20**

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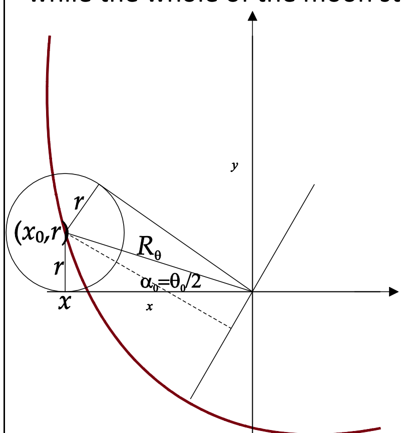
Characterize the coordinates of the moon when it looks largest to the observer.

Let $P(x_0, y_0)$ be the point where it looks largest to the observer. Since the distance between the moon and the observer decreases as it moves from apogee towards perigee, then the moon looks largest to the observer at the planet when it closest to the planet while the whole of the moon still can be seen in full. Hence, $y_0 = r$.



$$y_0 = r = R(\pi - \alpha_0\alpha_0)\sin(\pi - \alpha_0) = R_0 \sin(\pi - \alpha_0)$$
$$x_0 = R_0 \cos(\pi - \alpha_0)$$

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Determine the coordinates of the point.

Let the coordinate be (x_0, r) .

$$9\left(\frac{x_0}{2} + \frac{\sqrt{3}r}{2} - 4\right)^2 + 25\left(-\frac{\sqrt{3}x_0}{2} + \frac{r}{2}\right)^2 = 225$$

Rewrite it as a quadratic equation in x_0.

$$9\left(x_0 + r\sqrt{3} - 8\right)^2 + 25\left(-x_0\sqrt{3} + r\right)^2 = 900$$
$$84(x_0)^2 + \left(-32\sqrt{3}r - 144\right)x_0 + \left(52r^2 - 144\sqrt{3}r + 576 = 900\right)$$

Solving for x, we get

$$x_0 = \frac{4r\sqrt{3} + 18}{21} \pm \frac{15\sqrt{-r^2 + 4r\sqrt{3} + 9}}{21}$$

Then choose the smaller one

$$x_0 = \frac{4r\sqrt{3} + 18}{21} - \frac{15\sqrt{-r^2 + 4r\sqrt{3} + 9}}{21}$$

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Find $\tan\frac{\theta}{2}$. Hence

$$\tan\frac{\theta_0}{2} = \tan\alpha_0 = \frac{r}{x_0} = \frac{21r}{4r\sqrt{3} + 18 - 15\sqrt{-r^2 + 4r\sqrt{3} + 9}}$$

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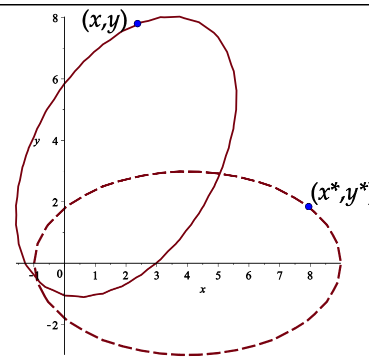
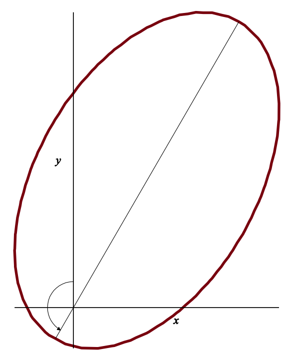
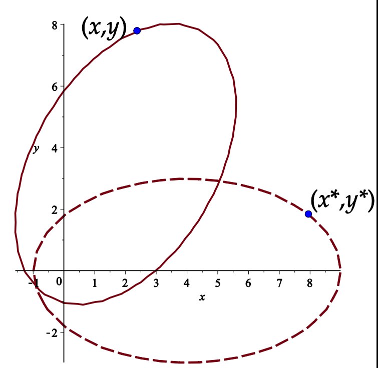
2. Cartesian version

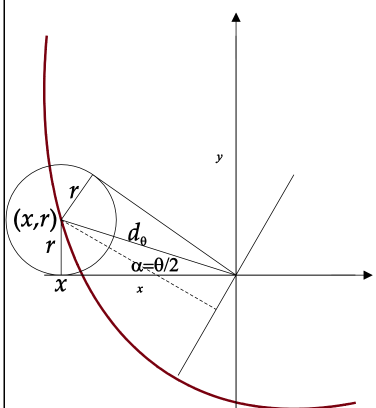
(a)	**Notice the standard version of the orbits** Simplify the equation by introducing new variables $x^* = \frac{x}{2} + \frac{\sqrt{3}y}{2} = x\cos\frac{\pi}{3} + y\sin\frac{\pi}{3}$ $y^* = -\frac{\sqrt{3}x}{2} + \frac{y}{2} = x\sin\left(-\frac{\pi}{3}\right) + y\cos\frac{\pi}{3}$ Write the orbit equation in terms new variables $9(x^* - 4)^2 + 25(y^*) = 225$  	10 10
	Determine semi major and semi minor axis of the orbit Since $9(x^* - 4)^2 + 25(y^*)^2 = 225$ is equivalent to $\frac{(x^* - 4)^2}{25} + \frac{(y^*)^2}{9} =$ The semi major axis is 5 The semi major axis is 3	5 5
(b)	**Determine the zenith angle at perigee** Since the original ellipse may be obtained from standard ellipse $\frac{(x^* - 4)^2}{25} + \frac{(y^*)^2}{9} = 1$ by $\pi/3$ counterclockwise rotation respect to the origin, the the zenith angle at perigee is equal to $\pi + \frac{\pi}{3} - \frac{\pi}{2} = \frac{5\pi}{6}$  	20
(c)	**Characterize the coordinates of the moon when it looks largest to the observer.** Let $P(x_0, y_0)$ be the point where it looks largest to the observer. Since the distance between the moon and the observer decreases as it moves from apogee towards perigee, then the moon looks largest to the observer at the planet when it closest to the planet while the whole of the moon still can be seen in full. Hence, $y_0 = r$.	10
	Find x_0. $9\left(\frac{x_0}{2} + \frac{\sqrt{3}r}{2} - 4\right)^2 + 25\left(-\frac{\sqrt{3}x_0}{2} + \frac{r}{2}\right)^2 = 225$ Rewrite it as a quadratic equation in x. $9\left(x_0 + r\sqrt{3} - 8\right)^2 + 25\left(-x_0\sqrt{3} + r\right)^2 = 900$ $\left(9x_0^2 + \left(18\sqrt{3}r - 144\right)x_0 + 27r^2 - 144\sqrt{3}r + 576\right) + \left(75x_0^2 - 50\sqrt{3}rx_0 + 25r^2\right) = 900$	

	$84(x_0)^2+\left(-32\sqrt{3}r-144\right)x_0+\left(52r^2-144\sqrt{3}r+576\right)=900$ Solving it for x we get $x_0=\frac{4r\sqrt{3}+18}{21}\pm\frac{15\sqrt{-r^2+4r\sqrt{3}+9}}{21}$ Then choose the smaller one $x_0=\frac{4r\sqrt{3}+18}{21}-\frac{15\sqrt{-r^2+4r\sqrt{3}+9}}{21}$	**20** **10**
	Find $\tan\frac{\theta}{2}$. Hence $\tan\frac{\theta}{2}=\tan\alpha_0=\frac{r}{x_0}=\frac{21r}{4r\sqrt{3}+18-15\sqrt{-r^2+4r\sqrt{3}+9}}$	**10**

3. Cartesian version

(a)	**Notice the standard version of the orbits** Simplify the equation by introducing new variables $x^*=\frac{x}{2}+\frac{\sqrt{3}y}{2}=x\cos\frac{\pi}{3}+y\sin\frac{\pi}{3}$ $y^*=-\frac{\sqrt{3}x}{2}+\frac{y}{2}=x\sin\left(-\frac{\pi}{3}\right)+y\cos\frac{\pi}{3}$ Write the orbit equation in terms new variables $9(x^*-4)^2+25(y^*)=225$	**10** **10**
	Determine semi major and semi minor axis of the orbit Since $9(x^*-4)^2+25(y^*)^2=225$ is equivalent to $\frac{(x^*-4)^2}{25}+\frac{(y^*)^2}{9}=$ The semi major axis is 5 The semi major axis is 3	 **5** **5**
(b)	**Notice the standard version of the orbits** Since the original ellipse may be obtained from standard ellipse $\frac{(x^*-4)^2}{25}+\frac{(y^*)^2}{9}=1$ by counterclockwise $\pi/3$ rotation respect to the origin, the the zenith angle at perigee is $\pi+\frac{\pi}{3}-\frac{\pi}{2}=\frac{5\pi}{6}$  	**20**

(c) **Identify and characterized the coordinates when the planet looks largest**.



Since the distance between the moon and the observer decreases as it moves from apogee towards perigee, then the moon looks largest to the observer at the planet when it closest to the planet while the whole of the moon still can be seen in full**.** Hence, it is at necessary that $y = r$. Let the coordinates be (x_0, r).

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Obtain expression for calculating distance from a point on an ellipse to the foci to set up an equation

Consider standard ellipse

$$\frac{(x^* - 4)^2}{5^2} + \frac{(y^*)^2}{3^2} = 1$$

Choose any point (x^*, y^*) on the ellipse. Let d_1 and d_2 be the distances from any point (x^*, y^*) on the ellipse to the foci $(0,0)$ and $d(8,0)$; $d_1^2 = (x^*)^2 + (y^*)^2$ and $d_2^2 = (x^* - 8)^2 + (y^*)^2$

Thus,

$$d_1^2 = (x^*)^2 + (y^*)^2 = (x^*)^2 + \left(9 - \frac{9(x^* - 4)^2}{5^2}\right) = \frac{(16(x^*)^2 + 72x^* + 81)}{25}$$

Therefore

$$d_1^2 = \frac{(16(x^*)^2 + 72x^* + 81)}{25} = x^2 + r^2$$

where $x^* = \frac{x}{2} + \frac{r\sqrt{3}}{2}$. Therefore, we can obtain the value of x by solving the above equation for x.

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Solve equation $\frac{(16(x^*)^2+72x^*+81)}{25} = x^2 + r^2$ **to obtain value of** x.

Substituting $x^* = \frac{x}{2} + \frac{r\sqrt{3}}{2}$ to the equation, we have

$$25d_1^2 = 25x^2 + 25r^2 = 16\left(\frac{x}{2} + \frac{r\sqrt{3}}{2}\right)^2 + 72\left(\frac{x}{2} + \frac{r\sqrt{3}}{2}\right) + 81 =$$

or

$$25x^2 + 25r^2 = 4x^2 + \left(8r\sqrt{3} + 36\right)x + \left(12r^2 + 36\sqrt{3}r + 81\right)$$

$$21x^2 - \left(8r\sqrt{3} + 36\right)x + \left(13r^2 - 36r\sqrt{3} - 81\right) = 0$$

Then use quadratic formula to obtain x in term of r. Choose smaller value of x to get larger value of $\tan\frac{\theta}{2} = \frac{r}{x}$. The smaller one is

$$x = \frac{4r\sqrt{3} + 18}{21} - \frac{15\sqrt{-r^2 + 4r\sqrt{3} + 9}}{21}$$

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	Find the expression of $\tan\frac{\theta}{2}$. Hence $$\tan\frac{\theta}{2}=\frac{r}{x}=\frac{21r}{4r\sqrt{3}+18-15\sqrt{-r^2+4r\sqrt{3}+9}}$$	**10**

2. Two massive stars A and B with masses m_A and m_B are separated by a distance d. Both stars orbit around their center of mass under gravitational force. Assume their orbits are circular and lie on the *X*-*Y* plane whose origin is at the stars' center of mass (see Figure 2)

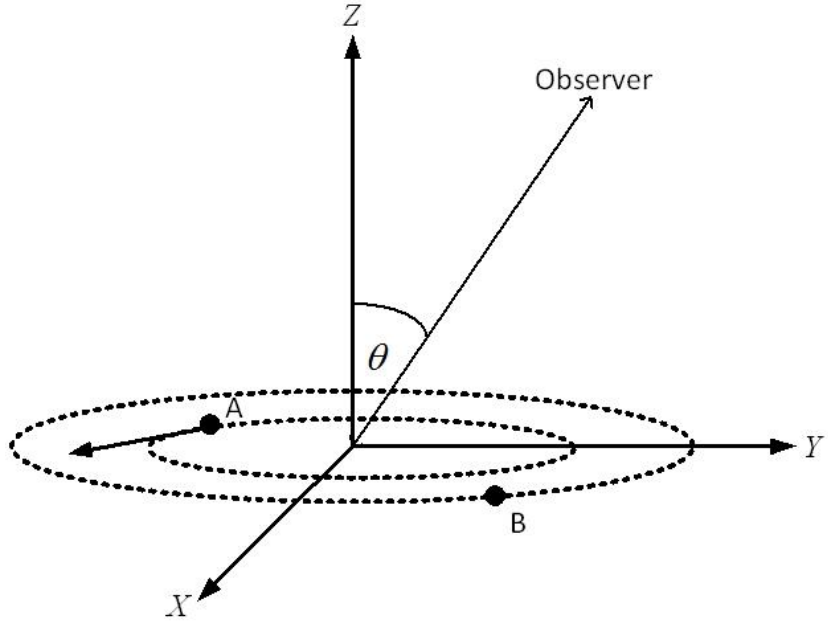

Figure 2

a. Find the expressions for the tangential and angular speeds of star A.

An observer standing on the *Y*-*Z* plane (see Figure 2) sees the stars from a large distance with an angle θ relatively to the *Z*-axis. He measures that the velocity component of star A to his line of sight has the form $K\ cos(\omega t + \varepsilon)$, where *K* and ε are positive.

b. Express $K^3/\omega G$ in terms of m_A, m_B , and θ where *G* is the universal gravitational constant.

Assume that the observer then identifies that star A has mass equal to $30M_S$ where M_S is the Sun's mass. In addition, he observes that star B produces X-rays and then realizes that it could be a neutron star or a black hole. This conclusion would depend on m_B, i.e.: i) If $m_B < 2M_S$, then B is a neutron star; ii) If $m_B > 2M_S$, then B is a black hole.

c. A measurement by the observer shows that $\frac{K^3}{\omega G} = \frac{1}{250} M_S$. In practice, the value of θ is usually not known. What is the condition on θ for star B to be a black hole?

Answer and Marking Scheme:

a.	The center of gravity of the stars is relatively to the star A given by $$r_A = \frac{m_B}{m_A+m_B}d$$ and since the orbit of A is a circle, then $$F_{AX} = \frac{Gm_Am_B}{d^2} = \frac{m_Av_A^2}{r_A}$$ So, we get $$v_A = m_B\sqrt{\frac{G}{(m_A+m_B)d}}$$ The angular velocity of A is given by $$\omega = \frac{v_A}{r_A} = \sqrt{\frac{G(m_A+m_B)}{d^3}}$$	**30**
b.	In Cartesian coordinate system, the velocity of A is $$\vec{v}_A(t) = v_A(-\sin(\omega t+\varepsilon)\,\hat{\imath} + \cos(\omega t+\varepsilon)\,\hat{\jmath})$$ Unit vector of the observer is $$\hat{r}_P = \cos\theta\,\hat{k} + \sin\theta\,\hat{\jmath}$$ so the component of $\vec{v}_A$ in the line of the observer sight is given by $$\vec{v}_A\cdot\hat{r}_P = v_A\sin\theta\ \cos(\omega t+\varepsilon)$$ Since the component of $\vec{v}_A$ in the line of the observer sight is $K\ cos(\omega t+\varepsilon)$, then $$K = v_A\sin\theta$$ Finally, we have $$\frac{K^3}{\omega G} = \frac{m_B^3}{(m_A+m_B)^2}\sin^3\theta$$ (**)	**30**
c.	From the result in b., namely eq. (**), we get $$\sin^3\theta = \frac{K^3}{\omega G}\frac{(m_A+m_B)^2}{m_B^3} < \frac{1}{250}\frac{32^2}{8} = \frac{64}{125}$$ Since $\theta \in [0,\pi]$, then $\sin\theta < 0{,}8$. Thus, the probability of B is a black hole is the same as the probability of $\sin\theta < 0{,}8$ for $\theta \in [0,\pi]$. So θ is less than 53^o or greater than 127^o.	**40**

3. Suppose a static spherical star consists of *N* neutral particles with radius *R* (see Figure 3).

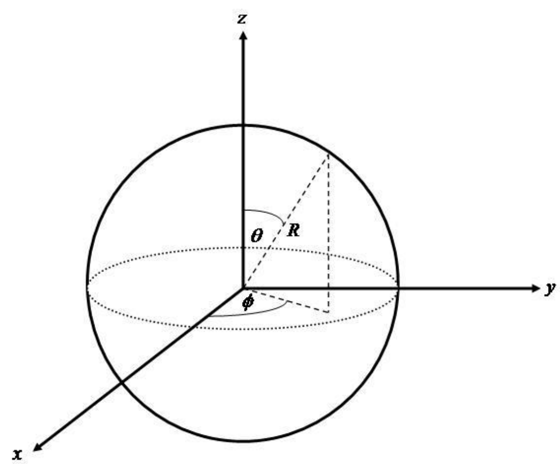

Figure 3

with $\mathbf{0 \le \theta \le \pi}$, $\mathbf{0 \le \phi \le 2\pi}$, satisfying the following equation of states

$$\boldsymbol{P\,V = N\,k\,\frac{T_R - T_0}{\ln(T_R/T_0)}} \tag{1}$$

where *P* and *V* are the pressure inside the star and the volume of the star respectively, *k* is the Boltzmann constant. $\boldsymbol{T_R}$ and $\boldsymbol{T_0}$ are the temperatures at the surface $\boldsymbol{r = R}$ and the temperature at the center $\boldsymbol{r = 0}$ respectively. Assume that $\boldsymbol{T_R \le T_0}$.

a. Simplify the stellar equation of state (1) if $\Delta\boldsymbol{T = T_R - T_0 \approx 0}$ (this is called ideal star)
(Hint: Use the approximation $\mathbf{ln}(\boldsymbol{1 + x}) \approx \boldsymbol{x}$ for small *x*)

Suppose the star undergoes a quasi-static process, in which it may slightly contract or expand, such that the above stellar equation of state (1) still holds.

The star satisfies first law of thermodynamics

$$\boldsymbol{Q = \Delta M c^2 + W} \tag{2}$$

where *Q*, *M*, and *W* are heat, mass of the star, and work respectively, while *c* is the light speed in the vacuum and $\boldsymbol{\Delta M \equiv M_{\text{final}} - M_{\text{initial}}}$.

In the following we assume $\boldsymbol{T_0}$ to be constant, while $\boldsymbol{T_R \equiv T}$ varies.

b. Find the heat capacity of the star at constant volume $\boldsymbol{C_v}$ in term of *M* and at constant pressure $\boldsymbol{C_p}$ expressed in $\boldsymbol{C_v}$ and *T* (Hint: Use the approximation $(\mathbf{1}+\boldsymbol{x})^{\boldsymbol{n}} \approx \mathbf{1}+\boldsymbol{nx}$ for small *x*)

Assuming that $\boldsymbol{C_v}$ is constant and the gas undergoes the isobaric process so the star produces the heat and radiates it outside to the space.

c. Find the heat produced by the isobaric process if the initial temperature and the final temperature are $\boldsymbol{T_i}$ and $\boldsymbol{T_f}$, respectively.

d.

For the next parts, assume the star is the Sun.

e. If the sunlight is monochromatic with frequency $\mathbf{5} \times \mathbf{10^{14}}$ Hz, estimate the number of photons radiated by the Sun per second.

f. Calculate the heat capacity $\boldsymbol{C_v}$ of the Sun assuming its surface temperature varies from 5500 K to 6000 K in one second.

Answer and Marking Scheme:

a.	Defining $\Delta T = T_R - T_0$ and $\Delta T \approx 0$, we have $$P\,V = N\,k\,\frac{\Delta T}{\ln(1+\Delta T/T_0)} \qquad (3)$$ using $\ln(1+\Delta T/T_0) \approx \Delta T/T_0$, we then obtain $$P\,V = N\,k\,T_0 \qquad (4)$$	**15**
b.	The internal energy of the star is $U = Mc^2$ ($U(T) = M(T)c^2$ for ideal star). Thus, the constant volume heat capacity of the star has the form $$C_v = \left(\frac{\Delta Q}{\Delta T}\right)_V = \left(\frac{\Delta M}{\Delta T}\right)_V c^2$$ for small ΔT. Then, using first law of thermodynamics, the constant pressure heat capacity of the star is $$C_p = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P\,\frac{\Delta V}{\Delta T} = C_v + P\,\frac{\Delta V}{\Delta T}$$ for small ΔT. Defining $\Delta T = T_2 - T_1$, then	**30**

	$$P\,\Delta V = N\,k\left(\frac{T_1 - T_0 + \Delta T}{\ln\big((T_1+\Delta T)/T_0\big)} - \frac{T_1 - T_0}{\ln(T_1/T_0)}\right)$$ Using the approximation $$\ln\big((T_1+\Delta T)/T_0\big) \approx \ln\left(\frac{T_1}{T_0}\right) + \frac{\Delta T}{T_1}$$ $$\left(1+\frac{\Delta T}{T_1 \ln(T_1/T_0)}\right)^{-1} \approx 1 - \frac{\Delta T}{T_1 \ln(T_1/T_0)}$$ then we have $$P\,\frac{\Delta V}{\Delta T} = \frac{N\,k}{\ln(T/T_0)}\left(1 - \frac{(T-T_0)/T}{\ln(T/T_0)}\right)$$ where $T_1 \equiv T$. Finally, we obtain $$C_p = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P\,\frac{\Delta V}{\Delta T} = C_v + \frac{N\,k}{\ln(T/T_0)}\left(1 - \frac{(T-T_0)/T}{\ln(T/T_0)}\right)$$	
c.	Since C_V is constant, the heat produced by the star is given by $$Q_H = C_v\,(T_f - T_i) + P\,\Delta\,V$$ $$= C_v\,(T_i - T_f) + N\,k\left(\frac{T_f - T_0}{\ln(T_f/T_0)} - \frac{T_i - T_0}{\ln(T_i/T_0)}\right)$$	**20**
d.		**15**
e.	Energy per second radiated by the Sun $$L_\odot = Nh\nu$$ where *N* is the number of photon. Thus $N = \frac{L_\odot}{h\,\nu} = \frac{3.96\times10^{26}}{6.6261\times10^{-34}\times\mathbf{5}\times\mathbf{10^{14}}} = 1.195\times10^{45}$ photons	**10**
f.	Energy per second radiated by the Sun is proportional to mass defect of the Sun $$L_\odot = \Delta M\,c^2$$ Thus, $$C_v = \frac{\Delta M\,c^2}{\Delta T} = \frac{L_\odot}{\Delta T} = \frac{3.96\times10^{26}}{6000-5500}\ \mathrm{J/K} = 7.92\times10^{23}\ \mathrm{J/K}$$	**10**