Long problem 1. Marking scheme - Eagles on the Caraiman Cross

1)	**10**
2)	**10**
B1)	10
B2)	10
C1)	5
C2)	5

1 The following notations are used: D_S the diameter of the Sun, d_{ES} Earth-Sun distance, θ angular diameter of the Sun as seen from the Earth:

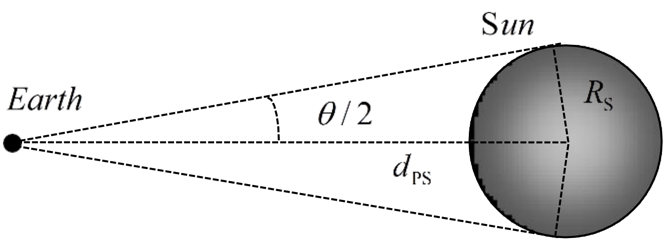


Fig. 1

According the fig. 1 the angular diameter of the Sun can be calculated as follows

$$\sin\frac{\theta}{2} = \frac{R_S}{d_{PS}} \approx \frac{\theta}{2};$$

$$\theta = \frac{2R_S}{d_{PS}} = \frac{D_S}{d_{PS}} = \frac{2\cdot 6{,}96\cdot 10^5\ \text{km}}{15\cdot 10^7\ \text{km}} = 0{,}00928\,\text{rad.}$$

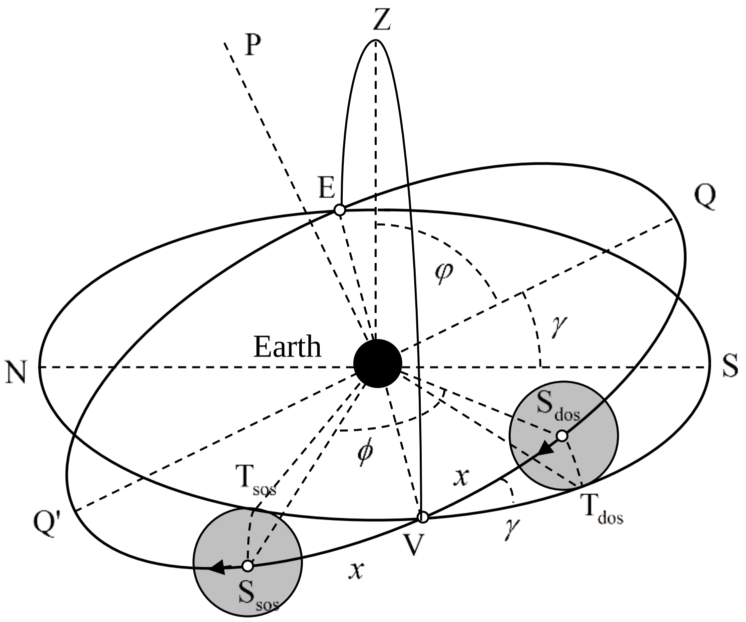
The figure 2 presents the Sun's evolution during sunset as seen by the astronomer. In an equinox day the Sun moves retrograde along the celestial equator. There are marked the following 3 positions of the Sun:

T_{dos} - The solar disc is tangent to the equatorial plane above the standard horizon – the start of the sunset;

S_{dos} - The center of the solar disc on the celestial equator in the moment of the sunset starts;

T_{sos} - The solar disc is tangent to the equatorial plane bellow the standard horizon – the end of the sunset

S_{sos} - The center of the solar disc on the celestial equator in the moment of the sunset ends;



The duration of the sunset is τ. During this time the center of the Sun moves along the equator from S_{dos} to S_{sos}. The vector-radius of the Sun rotates in equatorial plane with angle ϕ and in vertical plane with angle θ. i.e. the angular diameter of the Sun as seen from the Earth.

Considering that the Sun travels the distance $2x$ along the equatorial path with merely constant i.e. during time τ and that the spherical right triangle $S_{dos}T_{dos}V$ can be considered a plane one the following relations can be written:

$$\sin\gamma = \frac{R_S}{x};\ x = \frac{R_S}{\sin\gamma};\ \ 2x = \frac{2R_S}{\sin\gamma} = \frac{D_S}{\sin\gamma};$$

$$\tau = \frac{2x}{\text{v}} = \frac{2x}{\omega\cdot d_{PS}} = \frac{\frac{D_S}{\sin\gamma}}{\frac{2\pi}{T_P}\cdot d_{PS}} = \frac{\frac{D_S}{d_{PS}}}{\frac{2\pi}{T_P}\cdot\sin\gamma} = \frac{\theta\cdot T_P}{2\pi\cdot\sin\gamma};$$

$$\sin\gamma = \sin\left(90^\circ - \varphi\right) = \cos\varphi;$$

$$\tau = \frac{\theta\cdot T_P}{2\pi\cdot\cos\varphi};$$

$$\tau = \frac{0{,}00928\,\text{rad}\cdot 24\,\text{h}}{2\cdot 3{,}14\,\text{rad}\cdot\cos\left(45^\circ 21'\right)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}707}\,\text{min} \approx 3\,\text{min}.$$

2) If the atmospheric refraction is negligible the eagle on the top of the cross V_1 on figure 3 is on the same latitude (φ), as the astronomer but at the altitude H. Thus from the point of view of the V_1 the horizon line is below the standard horizon line with an angle $\Delta\alpha_1$,

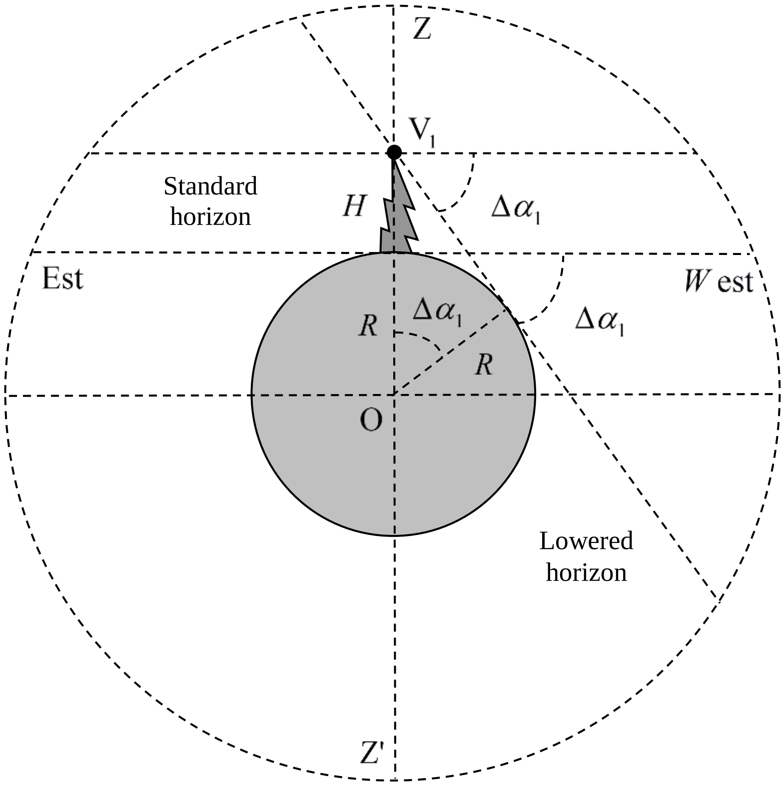


Fig. 3

$$\cos\Delta\alpha_1 = \frac{R}{R+H};$$

$$\sin\Delta\alpha_1 = \frac{\sqrt{(R+H)^2 - R^2}}{R+H} = \frac{\sqrt{2RH + H^2}}{R+H} \approx \frac{\sqrt{2RH}}{R} = \sqrt{\frac{2H}{R}} \approx \Delta\alpha_1;$$

$$\Delta\alpha_1 = \sqrt{\frac{2\cdot 2{,}3\,\text{km}}{6380\,\text{km}}} \approx 0{,}02685\,\text{rad} \approx 1{,}54^\circ.$$

For the observer V_1 the Sun will go below the lowered horizon after moving down under the standard horizon with angle $\Delta\alpha_1$ and moving along the equator with an angle $\Delta\beta_1$, as seen in fig. 4

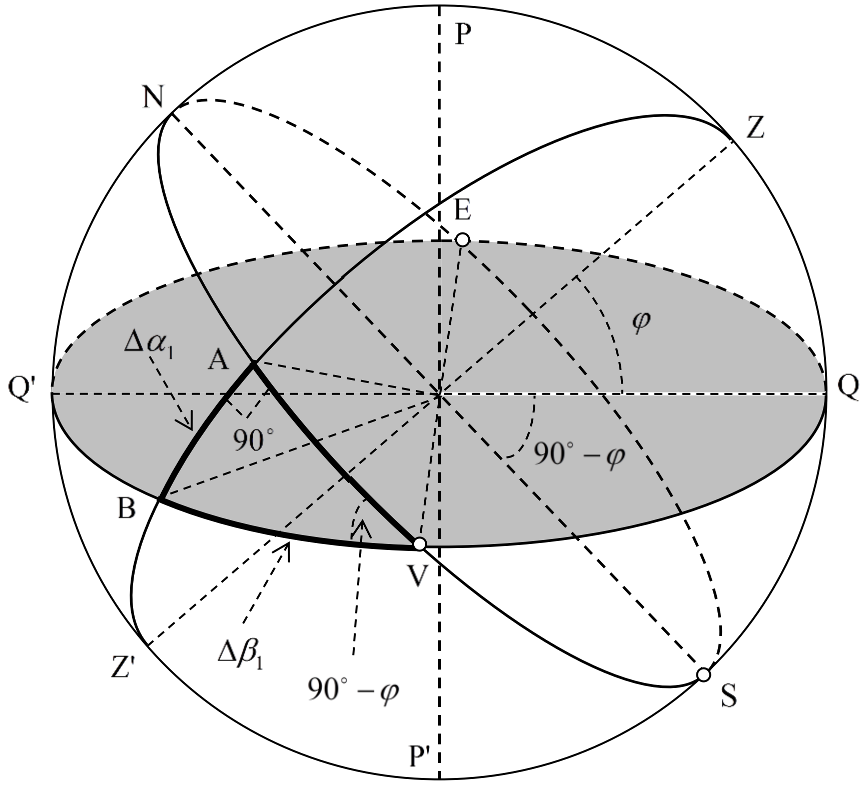


Fig.

In the right spherical triangle ABV by using the sinus formula :

$$\frac{\sin(90^\circ - \varphi)}{\sin \Delta\alpha_1} = \frac{\sin 90^\circ}{\sin \Delta\beta_1};$$

$$\frac{\cos\varphi}{\Delta\alpha_1} = \frac{1}{\Delta\beta_1}; \ \Delta\beta_1 = \frac{\Delta\alpha_1}{\cos\varphi};$$

$$\Delta\beta_1 = \omega \cdot \Delta\tau_1 = \frac{2\pi}{T_P} \cdot \Delta\tau_1;$$

$$\Delta\tau_1 = \frac{\Delta\alpha_1}{\cos\varphi} \cdot \frac{T_P}{2\pi} = \frac{1{,}54^\circ}{\cos(45^\circ 21')} \cdot \frac{24 \cdot 60 \text{ min}}{360^\circ} \approx 8{,}71 \text{minute},$$

Which represents the delay of the start of the sunset from the point of view of V_1 regardless to the astronomer due to the V_1 observer’s altitude.

The altitude effect on the total time of sunset can be calculated by using the fig. 5

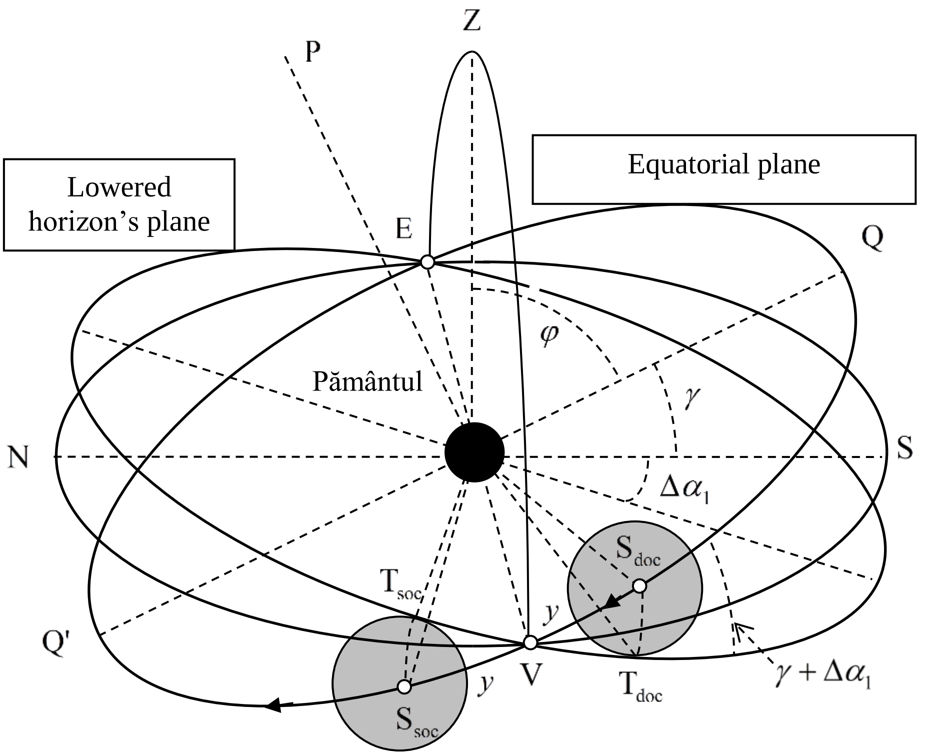


Fig.5

$$\sin(\gamma+\Delta\alpha_1)=\frac{R_S}{y};\ y=\frac{R_S}{\sin(\gamma+\Delta\alpha_1)};\ 2y=\frac{2R_S}{\sin(\gamma+\Delta\alpha_1)}=\frac{D_S}{\sin(\gamma+\Delta\alpha_1)};$$

$$\tau_1=\frac{2y}{\mathrm{v}}=\frac{2y}{\omega\cdot d_{PS}}=\frac{\dfrac{D_S}{\sin(\gamma+\Delta\alpha_1)}}{\dfrac{2\pi}{T_P}\cdot d_{PS}}=\frac{\dfrac{D_S}{d_{PS}}}{\dfrac{2\pi}{T_P}\cdot\sin(\gamma+\Delta\alpha_1)}=\frac{\theta\cdot T_P}{2\pi\cdot\sin(\gamma+\Delta\alpha_1)};$$

$$\gamma=90^\circ-\varphi;$$

$$\sin(\gamma+\Delta\alpha_1)=\sin(90^\circ-\varphi+\Delta\alpha_1)=\sin[90^\circ-(\varphi-\Delta\alpha_1)]=\cos(\varphi-\Delta\alpha_1);$$

$$\tau_1=\frac{\theta\cdot T_P}{2\pi\cdot\cos(\varphi-\Delta\alpha_1)};$$

$$\tau_1=\frac{0{,}00928\,\mathrm{rad}\cdot 24\,\mathrm{h}}{2\cdot 3{,}14\,\mathrm{rad}\cdot\cos(45^\circ-1{,}54^\circ)}=\frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}725}\,\mathrm{min}\approx 2{,}9350\,\mathrm{min},$$

Which represents the total duration of sunset for V_1 at altitude *H*.

Similarly for eagle V_2 at the same latitude (φ), but altitude $H+h$ (the top of the cross), the lowering effect on the horizon is measured by angle $\Delta\alpha_2$ thus

$$\cos\Delta\alpha_2=\frac{R}{R+H+h};$$

$$\sin\Delta\alpha_2 = \frac{\sqrt{(R+H+h)^2 - R^2}}{R+H+h} = \frac{\sqrt{2R(H+h)+(H+h)^2}}{R+H+h} \approx \frac{\sqrt{2R(H+h)}}{R} = \sqrt{\frac{2(H+h)}{R}} \approx \Delta\alpha_2;$$

$$\Delta\alpha_2 = \sqrt{\frac{2\cdot(2{,}3+0{,}0393)\,\text{km}}{6380\,\text{km}}} \approx 0{,}02707\,\text{rad} \approx 1{,}55^\circ;$$

$$\frac{\sin(90^\circ-\varphi)}{\sin\Delta\alpha_2} = \frac{\sin 90^\circ}{\sin\Delta\beta_2};$$

$$\frac{\cos\varphi}{\Delta\alpha_2} = \frac{1}{\Delta\beta_2};\;\; \Delta\beta_2 = \frac{\Delta\alpha_2}{\cos\varphi};$$

$$\Delta\beta_2 = \omega\cdot\Delta\tau_2 = \frac{2\pi}{T_P}\cdot\Delta\tau_2;$$

$$\Delta\tau_2 = \frac{\Delta\alpha_2}{\cos\varphi}\cdot\frac{T_P}{2\pi} = \frac{1{,}55^\circ}{\cos(45^\circ 21')}\cdot\frac{24\cdot 60\,\text{min}}{360^\circ} \approx 8{,}77\,\text{minute},$$

Which represents the delay of the start moment of the sunset for V_2 due to the altitude $H+h$.

Similar the total duration of the sunset for the observer V_2 :

$$\tau_2 = \frac{\theta\cdot T_P}{2\pi\cdot\cos(\varphi-\Delta\alpha_2)};$$

$$\tau_2 = \frac{0{,}00928\,\text{rad}\cdot 24\,\text{h}}{2\cdot 3{,}14\,\text{rad}\cdot\cos(45^\circ-1{,}55^\circ)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}726}\,\text{min} \approx 2{,}9309\,\text{min},$$

We may note the following:

- the horizon- lowering $\Delta\alpha$ is increased by the increase of the altitude;

$$(H < H+h \rightarrow \Delta\alpha_1 < \Delta\alpha_2;\, H\uparrow \rightarrow \Delta\alpha\uparrow).$$

- the delay of the moment of sunset start is increased by the increase of the altitude:

$$(H < H+h \rightarrow \Delta\tau_1 < \Delta\tau_2;\, H\uparrow \rightarrow \Delta\tau\uparrow).$$

- the total duration of sunset is reduced by the increase of the altitude:

$$(0 < H < H+h \rightarrow \tau > \tau_1 > \tau_2;\, H\uparrow \rightarrow \tau\downarrow).$$

Conclusions:

If we consider t_0 the moment of sunset star for the astronomer

- for V_1 the sunset starts at $t_0 + 8{,}71\,\text{min}$ and ends at $t_0 + 8{,}71\,\text{min} + 2{,}9350\,\text{min} = t_0 + 11{,}6450\,\text{min}$
- for V_2 the sunset starts at $t_0 + 8{,}77\,\text{min}$ and ends at $t_0 + 8{,}77\,\text{min} + 2{,}9309\,\text{min} = t_0 + 11{,}7009\,\text{min}$
- Thus eagle from the plateau leaves first the cross;
- The time between the leaving moments is:

$$\Delta t = t_0 + 11{,}7009\,\text{min} - t_0 - 11{,}6450\,\text{min} = 0{,}0559\,\text{min} = 3{,}354\,\text{s}.$$

b)

As seen in fig. 6 the length of the cross on the plateau will be minimum when the Sun passes the local meridian, i.e. the height of the Sun above the horizon will be maximum:

$(h_{max} = \gamma = 90^\circ - \varphi)$.

Thus the shadow of the horizontal arms of the cross is superposed on the shadow of the vertical pillow.

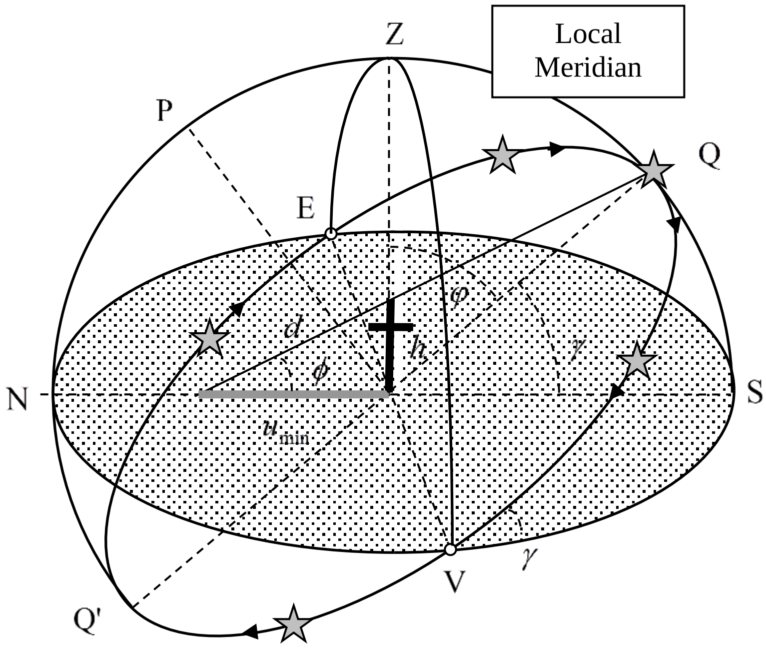


Fig. 6

In this conditions :

$$\sin\phi = \frac{h}{d};\ \phi \approx \gamma = 90^\circ - \varphi;$$

$$d = \frac{h}{\sin\phi} \approx \frac{h}{\sin\gamma} = \frac{h}{\sin(90^\circ - \varphi)} = \frac{h}{\cos\varphi} = \frac{39{,}3\,\mathrm{m}}{\cos 45^\circ} = \frac{39{,}3}{0{,}707}\,\mathrm{m} \approx 55{,}58\,\mathrm{m};$$

The distance between the two eagles is

$$u_{\min} = h\cdot\cot\phi \approx h\cdot\cot\varphi = h\cdot\cot(90^\circ - \varphi) = h\cdot\tan\varphi = 39{,}3\,\mathrm{m}.$$

2) In the above mentioned conditions the shadow of the arm oriented toward South is on the vertical pillow of the cross, as seen in fig. 7:

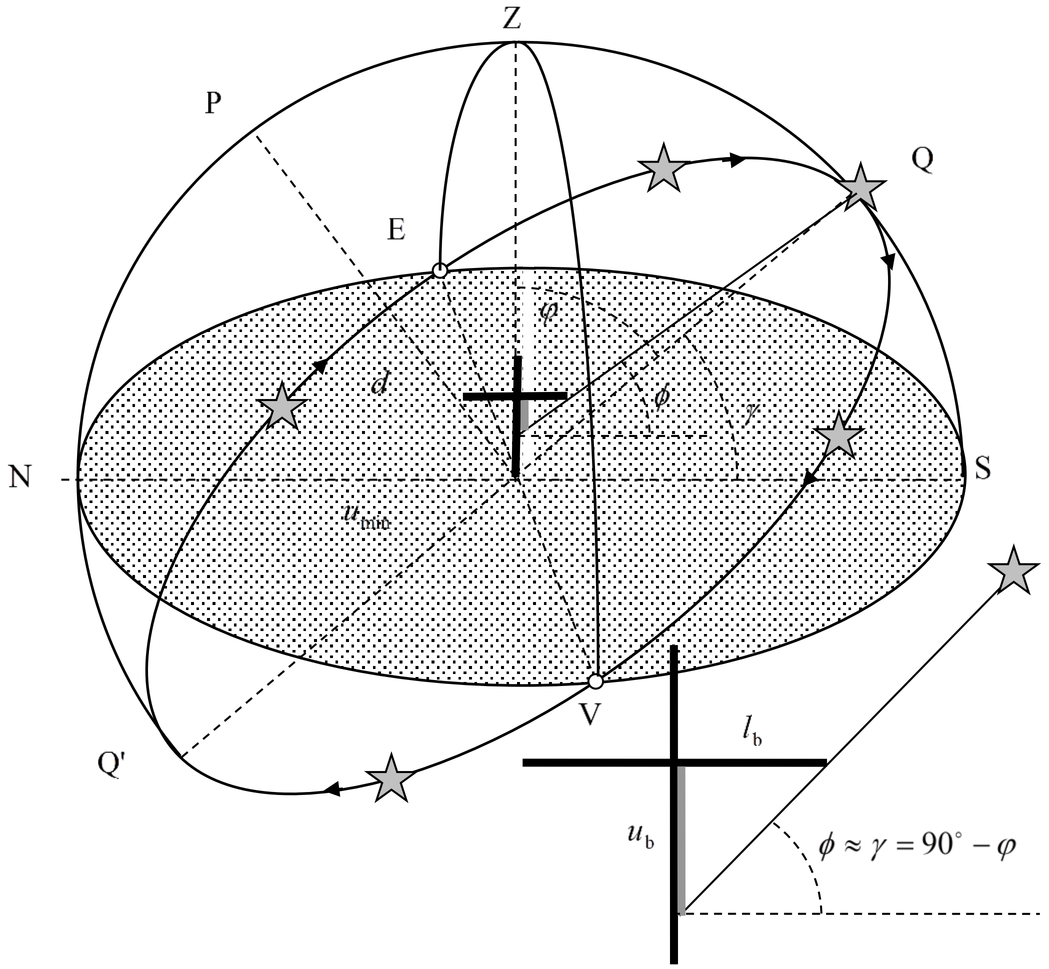


Fig. 7

$$\tan\varphi = \frac{l_b}{u_b};\ l_b = u_b \cdot \tan\varphi = 7\ \mathrm{m} \cdot \tan 45^\circ = 7\ \mathrm{m},$$

Which represents the length of the cross arm.

C)

1) For an observer situated in the center O of the celestial topocentric sphere, at latitude φ, at sea level, al the stars are circumpolar ones see fig. 8. Their diurnal parallels, parallel with the equatorial parallel, are above the real local horizon (N_0S_0). The star σ_0 is at the circumpolar limit because its parallel touches the real local horizon in point N_0 but still remains above it. Thus σ_0 is a marginal circumpolar star. Without taking into account the atmospheric refraction:

From the isosceles triangle $O\sigma_0 N_0$ results the σ_0 declination:

$$\delta_{0,\min} + 90^\circ + (\varphi - \delta_{\min}) = 180^\circ;$$

$$\delta_{0,\min} = 90^\circ - \varphi.$$

For the observer at latitude φ, but at height h, taking into account the effect of lowering of the horizon the star σ'' will meet the problem requirements see figure 10. The new horizon is $N''S''$ and declination of σ'' is $\delta''_{\min} < \delta_{0\min}$. Star σ_0 remains a circumpolar one but above the limit.

From the isosceles triangle $N''O\sigma''$ the declination of star σ'' will be:

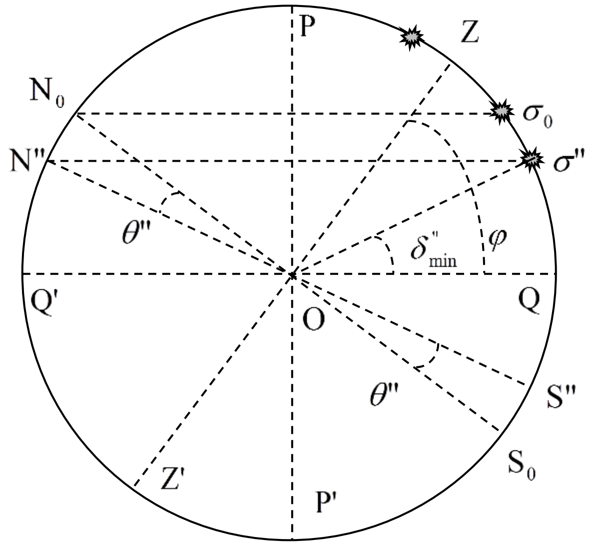


Fig. 10

$$2\delta''_{\min} + (\varphi - \delta''_{\min}) + (90^\circ + \theta'') = 180^\circ;$$
$$\delta''_{\min} = 90^\circ - \varphi - \theta''.$$

By taking into account the refraction effect and the altitude effect, from triangle $NO\sigma$ in figure 11, the declination will be

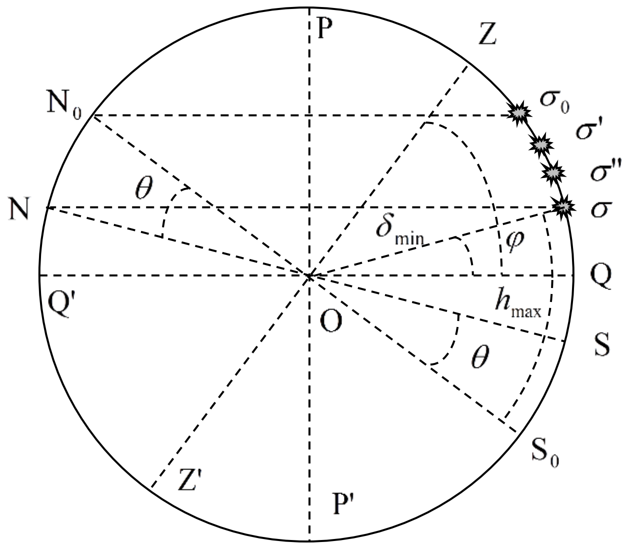


Fig. 11

$$2\delta_{\min} + (\varphi - \delta_{\min}) + (90^\circ + \theta) = 180^\circ;$$
$$\theta = \theta' + \theta''; \ \theta' = \xi = 34': \ \theta'' = \Delta\alpha_2 = 1{,}55^\circ;$$

$$\delta_{min} = 90^o - \varphi - \theta' - \theta'' = 90^o - \varphi - \theta' - \Delta\alpha_2;$$

$$\delta_{min} = 90^o - 45^o - 0{,}56^o - 1{,}55^o \approx 42{,}9^o.$$

1) The maximum height above the horizon will be

$$h_{max} = 90^o + \delta_{min} - \varphi = 90^o + 42{,}9^o - 45^o = 87{,}9^o.$$

Long problem 2. Marking scheme - From Romania …. to Antipod! … a ballistic messenger

a)	8
b)	8
c)	8
d)	8
e)	8
f)	10

a) The two places are represented in the figure.

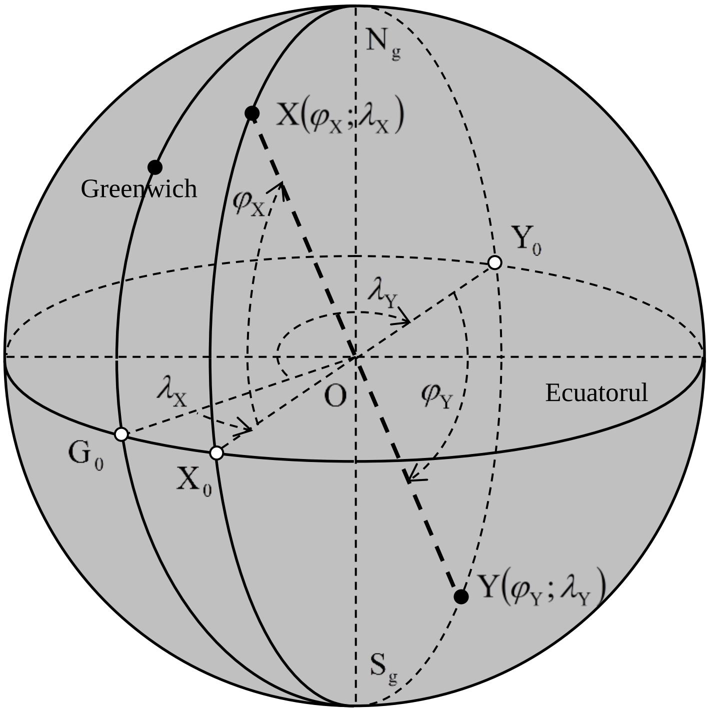


Fig.

$$\varphi_Y = \varphi_{Y,Sud} = \varphi_{X,Nord} = \varphi_X;$$

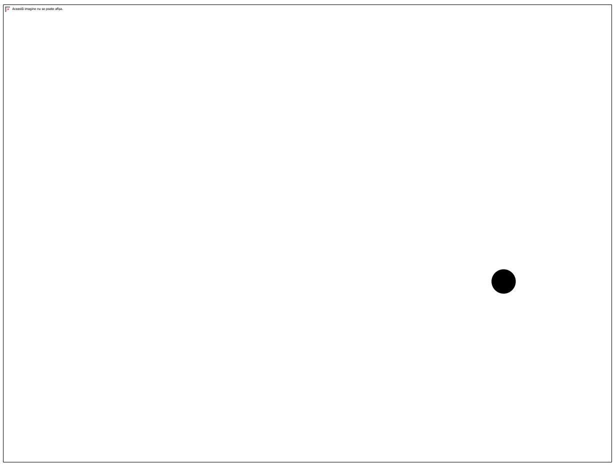
$$\lambda_{Y,Vest} + \lambda_{X,Eest} = 180^\circ;\ \lambda_Y + \lambda_X = 180^\circ.$$

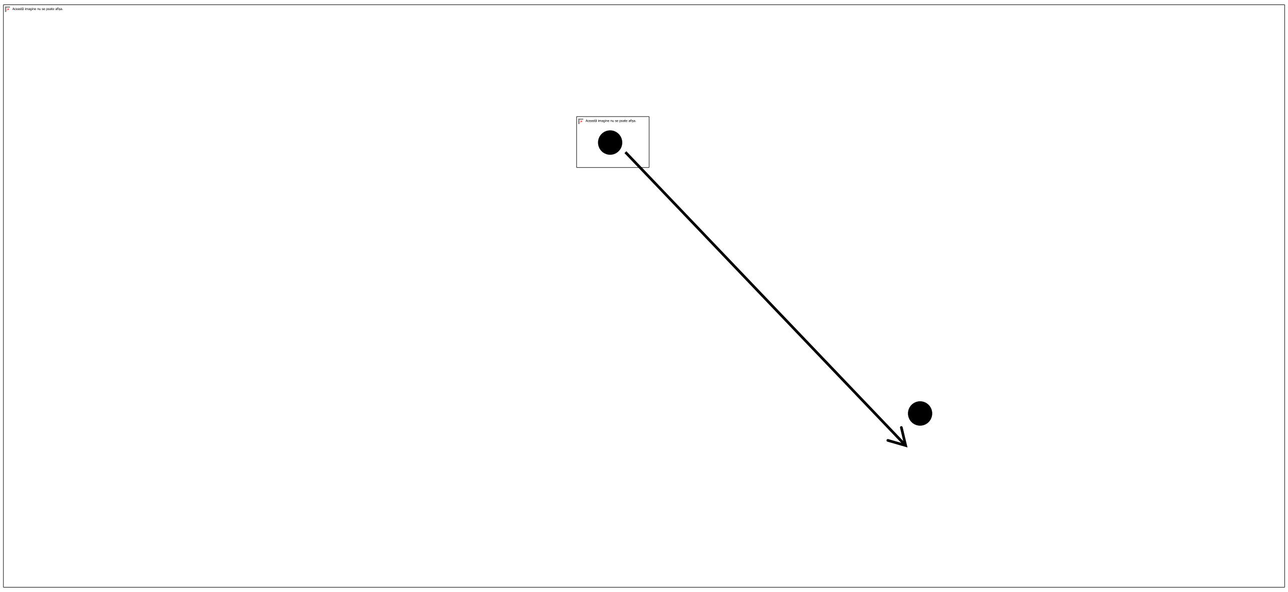
$$\varphi_{Romania} = 43^\circ \text{ Nord};\ \lambda_{Romania} = 30^\circ \text{ Est},$$

The landing point is

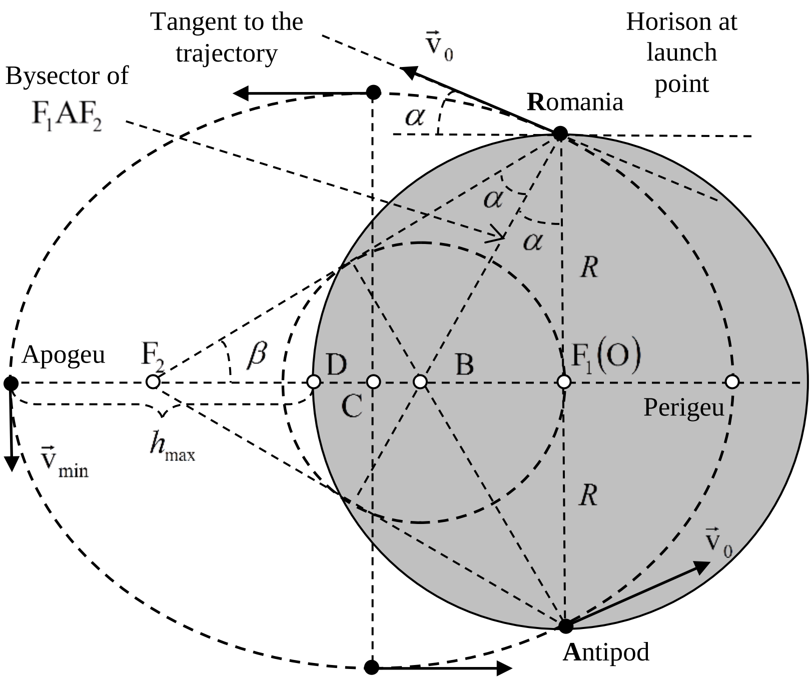
$$\varphi_{Antipod} = 43^\circ \text{ Sud};\ \lambda_{Antipod} = 150^\circ \text{ Vest},$$

Somewhere South EEst from Tasmania (South from Australia).





b) The schetch of the trajectory



In order to hit the point the trajectory of the missile has to be an elypse with the Earth center in the center of the Earth. *Se știe că*:

$$F_2B = 2 \cdot F_1B;\ F_1F_2 = 3 \cdot F_1B.$$

Rezultă:

$$\tan 2\alpha = \frac{F_1F_2}{R};\ F_1F_2 = R \cdot \tan 2\alpha;$$

$$\tan \alpha = \frac{F_1B}{R};\ \ F_1B = R \cdot \tan \alpha;$$

$$R \cdot \tan 2\alpha = F_1B = R \cdot \tan \alpha;$$

$$\tan 2\alpha = 3 \cdot \tan \alpha;$$

$$\frac{\sin 2\alpha}{\cos 2\alpha} = 3\frac{\sin \alpha}{\cos \alpha};$$

$$\frac{2\sin \alpha \cdot \cos \alpha}{\cos 2\alpha} = 3\frac{\sin \alpha}{\cos \alpha};$$

$$2\cos^2 \alpha = 3\cos 2\alpha;\ \ 2\cos^2 \alpha = 3\left(\cos^2 \alpha - \sin^2 \alpha\right);$$

$$3\sin^2 \alpha = \cos^2 \alpha;\ \tan^2 \alpha = \frac{1}{3};$$

$$\tan \alpha = \frac{\sqrt{3}}{3};\ \ \alpha = 30^\circ;\ \ 2\alpha = 60^\circ;\ \beta = 90^\circ - 2\alpha = 30^\circ;\ 2\beta = 60^\circ;$$

$$\Delta\left(RF_2A\right) \rightarrow \text{triunghi echilateral};$$

$$RF_2 = AF_2 = RA = 2R;$$

$$RF_2 + RF_1 = 2a = 3R;$$

$$a = \frac{3}{2}R;$$

$$\mathrm{v}_0 = \sqrt{KM\left(\frac{2}{r} - \frac{1}{a}\right)};\ r = R;\ g_0 = K\frac{M}{R^2};$$

$$\mathrm{v}_0 = \sqrt{\frac{KM}{R^2} \cdot R^2\left(\frac{2}{R} - \frac{2}{3R}\right)} = 2\sqrt{\frac{g_0 R}{3}}.$$

c)

$$\mathrm{v}_{\mathrm{Antipod}} = \mathrm{v}_0 .$$

d)

$$\mathrm{F_1F_2} = R \cdot \tan 2\alpha = 2c;\ c = \frac{R}{2} \cdot \tan 2\alpha = \frac{R}{2} \cdot \tan 60^\circ = \frac{\sqrt{3}}{2}R;$$

$$b = \sqrt{a^2 - c^2} = \sqrt{\frac{3}{2}}R;$$

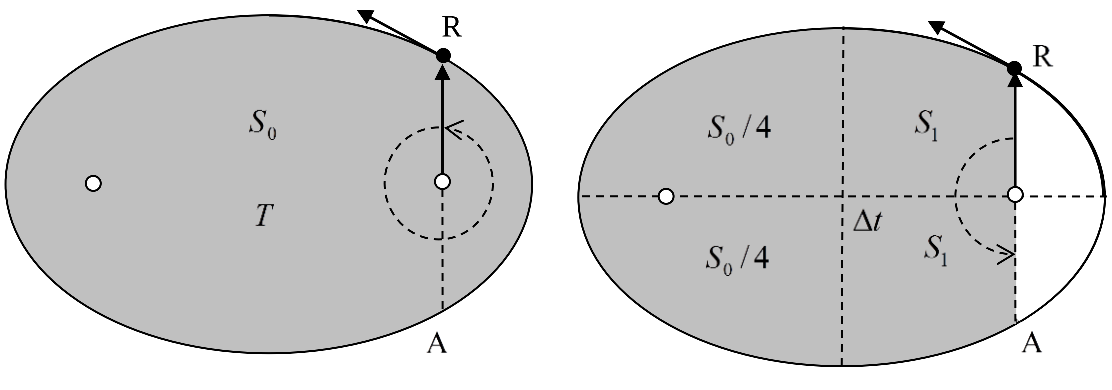
$$2a = 2r_{\min} + 2c;\ r_{\min} = a - c = \frac{1}{2}\left(3 - \sqrt{3}\right)R;$$

$$r_{\max} = 2a - r_{\min} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$\mathrm{v}_{\min} = \sqrt{KM\left(\frac{2}{r_{\max}} - \frac{1}{a}\right)};\ r_{\max} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$\mathrm{v}_{\min} = \sqrt{\frac{KM}{R^2} \cdot R^2\left(\frac{4}{\left(3+\sqrt{3}\right)R} - \frac{2}{3R}\right)} = \sqrt{\frac{2g_0R}{3} \cdot \frac{3-\sqrt{3}}{3+\sqrt{3}}}.$$

e) Accordin to Kepplers laws:



$$\Omega = \frac{\mathrm{d}S}{\mathrm{d}t} = \mathrm{constant};$$

$$\frac{S_0}{T} = \frac{2\frac{S_0}{4} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{\frac{S_0}{2} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{S_0 + 4S_1}{2 \cdot \Delta t};$$

$$S_0 = \pi ab;\ S_1 = \frac{ab}{2}\left[\sqrt{1-\frac{b^2}{a^2}}\cdot\frac{b}{a} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right];$$

$$\Delta t = \frac{S_0 + 4S_1}{2S_0}\cdot T = \left(\frac{1}{2} + 2\frac{S_1}{S_0}\right)\cdot T;$$

$$T = 2\pi\sqrt{\frac{a^3}{KM}};\ T = \frac{2\pi}{R}\sqrt{\frac{a^3}{g_0}};$$

$$\frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot\sqrt{1-\frac{b^2}{a^2}} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right);$$

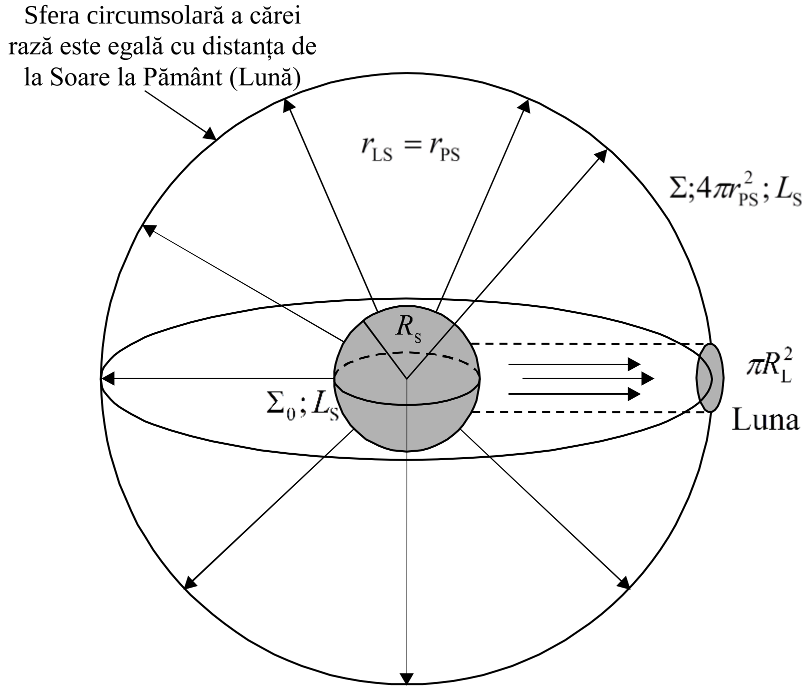
$$\sqrt{1-\frac{b^2}{a^2}} = e;\ \frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot e + \arcsin e\right);$$

$$\Delta t = \left(\frac{1}{2} + \frac{eb}{\pi a} + \frac{\arcsin e}{\pi}\right)\cdot T.$$

f) The integral luminosity of Sun:

$$L_{\mathrm{S}} = \frac{E_{\mathrm{emis,Soare}}}{t} = 3{,}86\cdot 10^{26}\ \mathrm{W},$$

Dacă For a circumsolar surface Σ with radius r_{PS}, see picture bellow the solar radiation enegy passing through the surface in one second is L_S.



Density of solar flux

$$\phi_{Soare, r_{PS}} = \frac{E_{emis,Soare}}{St} = \frac{\frac{E_{emis,Soare}}{t}}{S} = \frac{L_S}{S} = \frac{L_S}{4\pi r_{PS}^2} = \text{constant}.$$

$$F_{incident, FullMoon} = \phi_{Sun, r_{PS}} \cdot \pi R_L^2.$$

Dacă α_L este albedoul Lunii, rezultă:

$$\alpha_{\mathrm{L}} = \frac{F_{\text{reflectat}, FullMoon}}{F_{\text{incident}, FullMoon}},$$

unde $F_{\text{reflectat, Luna Plina}}$ – fluxul energetic al radiațiilor reflectate de Luna Plină spre observatorul de pe Pământ;

$$F_{\text{reflectat}, FullMoon} = \alpha_{\mathrm{L}} \cdot F_{\text{incident}, FullMoon} = \alpha_{\mathrm{L}} \cdot \phi_{\text{Soa,,re}, r_{\mathrm{PS}}} \cdot \pi R_{\mathrm{L}}^2.$$

În consecință, densitatea fluxului energetic ajuns la observator, după reflexia pe suprafața Lunii, este:

$$\phi_{moon,\ \text{observator}} = \frac{F_{\text{reflectat},\ FullMoon}}{2\pi r_{\mathrm{PL}}^2} = \alpha_{\mathrm{L}} \cdot \phi_{\text{Soare}, r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{L}}^2}{2\pi r_{\mathrm{PL}}^2}.$$

Symilarly

$$\phi_{\text{proiectil, observator}} = \frac{F_{\text{reflectat, proiectil}}}{4\pi r_{\text{D,proiectil}}^2} = \alpha_{\text{proiectil}} \cdot \phi_{\text{Soare}, r_{\mathrm{PS}}} \cdot \frac{\pi R_{\text{proiectil}}^2}{4\pi r_{\text{D,proiectil}}^2}.$$

În expresia anterioară s-a avut în vedere faptul că densitatea fluxului energetic al proiectilului la observator rezultă din distribuirea prin suprafața sferei cu raza $r_{\text{P,proiectil}}$.

Utilizând formula lui Pogson, vom compara magnitudinea aparentă vizuală a Lunii Pline cu magnitudinea aparentă vizuală a proiectilului balistic:

$$\log \frac{\phi_{\text{Luna,observator}}}{\phi_{\text{proiectil,observator}}} = -0{,}4\left(m_{\text{Luna Plina}} - m_{\text{proiectil}}\right);$$

$$\log \frac{\phi_{\text{Luna,observator}}}{\phi_{\text{proiectil,observator}}} = \log \frac{\alpha_{\mathrm{L}} \cdot \phi_{\text{Soare}, r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{L}}^2}{2\pi r_{\mathrm{PL}}^2}}{\alpha_{\text{proiectil}} \cdot \phi_{\text{Soare}, r_{\mathrm{PS}}} \cdot \frac{\pi R_{\text{proiectil}}^2}{4\pi r_{\text{D,proiectil}}^2}} = \log \frac{\alpha_{\mathrm{L}} \cdot \frac{R_{\mathrm{L}}^2}{r_{\mathrm{PL}}^2}}{\alpha_{\text{proiectil}} \cdot \frac{R_{\text{proiectil}}^2}{2 r_{\text{D,proiectil}}^2}};$$

$$\log \frac{\alpha_{\mathrm{L}} \cdot \frac{R_{\mathrm{L}}^2}{r_{\mathrm{PL}}^2}}{\alpha_{\text{proiectil}} \cdot \frac{R_{\text{proiectil}}^2}{2 r_{\text{D,proiectil}}^2}} = -0{,}4\left(m_{\mathrm{L}} - m_{\text{proiectil}}\right);$$

$$\log \frac{\alpha_{\mathrm{L}}}{\alpha_{\text{proiectil}}} \cdot \left(\frac{R_{\mathrm{L}}}{R_{\text{proiectil}}}\right)^2 \cdot 2 \cdot \left(\frac{r_{\text{D,proiectil}}}{r_{\mathrm{PL}}}\right)^2 = -0{,}4\left(m_{\mathrm{L}} - m_{\text{proiectil}}\right);$$

$$\alpha_{\mathrm{L}} = 0{,}12;\ \alpha_{\text{proiectil}} = 1;$$

$$R_{\mathrm{L}} = 1738\ \text{km};\ R_{\text{proiectil}} = 400\ \text{mm};$$

$$r_{\text{D,proiectil}} = r_{\text{max,observator–proiectil}} = h_{\max} = r_{\max} - R;\ r_{\max} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$h_{\max} = \frac{1}{2}\left(3 + \sqrt{3}\right)R - R = \frac{1}{2}\left(1 + \sqrt{3}\right)R \approx 8700\ \text{km};$$

$$r_{PL} = r_{observator,Luna} = 384400\,\text{km};\ m_L = -12{,}7^m;$$

$$\log\frac{\alpha_L}{\alpha_{proiectil}} + 2\log\frac{R_L}{R_{proiectil}} + \log 2 + 2\log\frac{r_{D-proiectil}}{r_{PL}} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000\,\text{m}}{0{,}400\,\text{m}} + \log 2 + 2\log\frac{8700\,\text{km}}{384400\,\text{km}} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000}{0{,}400} + \log 2 + 2\log\frac{8700}{384400} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$-0{,}920818754 + 13{,}27597956 + 0{,}301029995 - 3{,}290528253 = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$23{,}4^m = 12{,}7^m + m_{proiectil};$$

$$m_{proiectil} = 10{,}7^m;$$

$$m_{max} \approx 6^m;\ m_{proiectil} > m_{max},$$

The projectile wasn't seen when it was at its apogee