Theoretical Exam - Long Questions

Question 1

In a homogeneous and isotropic universe, the matter (baryonic matter + dark matter) density parameter $\Omega_m = \frac{\rho_m}{\rho_c}$ = 32%, where ρ_m is the matter density and ρ_c is the critical density of the Universe.

(1) Calculate the average matter density in our local neighbourhood.
(2) Calculate the escape velocity of a galaxy 100 Mpc away from us. Assume that the recession velocity of galaxies in Hubble's law equals the corresponding escape velocity at that distance, for the critical density of the Universe that we observe.
(3) The particular galaxy is orbiting around the centre of our cluster of galaxies on a circular orbit. What is the angular velocity of this galaxy on the sky?
(4) Will we ever discriminate two such galaxies that are initially at the same line of sight, if they are both moving on circular orbits but at different radii (answer "Yes" or "No")? [Assume that the Earth is located at the centre of our local cluster.]

Answer:

(1) The critical density $\rho_c = \frac{3H_0^2}{8\pi G}$ **(9 points)**

If the matter density parameter $\Omega_m = \frac{\rho_m}{\rho_c}$ is 32%, thus $\rho_m = 0.32\frac{3H_0^2}{8\pi G}$. **(3 points)**

From the latest estimate of $H_0 = 67.8\ \frac{km\,s^{-1}}{Mpc}$ we obtain $\rho_m = 8.6\times10^{-27}$ kg m^{-3} **(4 points)**

(2) The escape velocity is $\upsilon_{esc} = \sqrt{\frac{2GM}{d}}$. **(4 points)**

By replacing $M = \rho_c \frac{4\pi}{3} d^3$ **(2 points)**

we obtain, for the escape velocity within d = 100 Mpc

$$\upsilon_{esc} = \sqrt{\frac{8\pi}{3} G \times 0.32 \times \frac{3\mathrm{H}_0^2}{8\pi G}} \times 100\,\mathrm{M}pc = 3835\ \mathrm{km\ s^{-1}}$$ **(8 points)**

(3) If a galaxy is orbiting around the centre of our galaxy, its velocity is $\frac{1}{\sqrt{2}}$ of its escape velocity. Thus

$$\omega = \frac{v}{d} = \frac{v_{esc}/\sqrt{2}}{d} = \frac{H_0 d\sqrt{\Omega_m/2}}{d} = \frac{\sqrt{0.32}\,(67.8\times10^3 m/s)/(3.09\times10^{22} m)}{\sqrt{2}} = 8.8\times10^{-19} rad/s$$ **(12 Points)**

This is

$1.8\times10^{-13} arc\sec/s$ and it does not depend on the distance d.

(4) Therefore we will never be able to resolve them and the answer is "No". **(8 Points)**

Question 2

A spacecraft is orbiting the Near Earth Asteroid (2608) *Seneca* (staying continuously very close to the asteroid), transmitting pulsed data to the *Earth*. Due to the relative motion of the two bodies (the asteroid and the Earth) around the *Sun*, the time it takes for a pulse to arrive at the ground station varies approximately between 2 and 39 minutes. The orbits of the Earth and *Seneca* are coplanar. Assuming that the *Earth* moves around the *Sun* on a circular orbit (with radius a_{Earth} = 1 AU and period T_{Earth} = 1 yr) and that the orbit of *Seneca* does not intersect the orbit of the *Earth*, calculate:
(1) the semi-major axis, a_{Sen} the eccentricity, e_{Sen} of *Seneca's* orbit around the *Sun*
(2) the period of *Seneca's* orbit, T_{Sen} and the average period between two consecutive oppositions, T_{syn} of the *Earth-Seneca* couple
(3) an approximate value for the mass of the planet Jupiter, M_{Jup} (assuming this is the only planet of our Solar system with non-negligible mass compared to the Sun). Assume that the presence of Jupiter does not influence the orbit of Seneca.

Answer :
(1) For Δt_b = 2 min = 120 sec, the distance travelled by a light pulse is $R_1 = c\times\Delta t$ or R_1 = 0.24 AU, while for Δt_a = 39 min the maximum distance is $R_2 = c\times\Delta t$ or R_2 = 4.67 AU. **(4 points)**

Since the orbits do not intersect and the R_2 exceeds by far 1 AU, the orbit of *Seneca* is exterior to that of the *Earth*. R_1 corresponds to the minimum relative distance of the two bodies (i.e. at opposition), while R_2 corresponds to the maximum relative distance (i.e. at conjunction).

If $q = a\times(1–e)$ is the perihelion and $Q = a\times(1+e)$ the aphelion distance of *Seneca*, then $R_1 = q–1$ AU (the minimum distance of *Seneca* from the *Sun* minus the semi-major axis of the *Earth's* orbit), while $R_2 = Q+1$ AU (the maximum distance of *Seneca* from the *Sun* plus the semi-major axis of the *Earth's* orbit). **(6 points)**
Thus,

$a_{\text{Sen}}(1–e_{\text{Sen}}) = 1+R_1$ = 1.24 AU
$a_{\text{Sen}}(1+e_{\text{Sen}}) = R_2–1$ = 3.67 AU from which one finds $\boldsymbol{a_{\text{Sen}}} \sim 2.46$ AU **(2 point)**
and $\boldsymbol{e_{\text{Sen}}} \sim 0.49$ **(2 points)**

(figures rounded to 2 dec. digits)

(2) The period, T_{Sen}, of *Seneca's* orbit can be found by using Kepler's 3rd law for *Seneca* and the *Earth* (ignoring their small masses, compared to the Sun's):

$a_{\text{Sen}}^3 / T_{\text{Sen}}^2 = a_{\text{Earth}}^3 / T_{\text{Earth}}^2 = 1$ **(6 points)**

from which one finds $\boldsymbol{T_{\text{Sen}}} \sim 3.87$ yr **(2 point)**

(Alternatively one can use Kepler's law for *Seneca* only and use natural units for the mass of the *Sun* and G, i.e. [mass] = 1 M_{Sun}, [t]= 1 yr and [r] = 1 AU, in which case $\boldsymbol{T_{\text{Sen}} = a_{\text{Sen}}^{3/2}}$ = 3.86 yr)
(8 points)

Assuming non-retrogate orbit, the synodic period, T_{syn} of *Seneca* and *Earth* is given by

$1/T_{syn} = 1/T_{Earh} - 1/T_{Sen}$ (*Seneca* is superior) **(6 points)**

which gives $\boldsymbol{T_{syn}}$ **=** 1.35 yr **(2 point)**

(3) From Kepler's law, we can calculate the mass of the *Sun*

$4\pi^2\, a_{Sen}{}^3 / T_{Sen}{}^2 = G\, M_{Sun}$, i.e. $M_{Sun} \sim 1.988\times10^{30}$ kg (depending on accuracy) **(4 points)**

Then, using Kepler's 3rd law for *Seneca* and *Jupiter*, one gets

$(a_{Jup}{}^3 / T_{Jup}{}^2) / (a_{Sen}{}^3 / T_{Sen}{}^2) = (M_{Sun} + M_{Jup}) / M_{Sun} = 1 + x$ **(4 points)**

where $x =$ is the mass ratio of *Jupiter* to the *Sun*. Then, solving for x we get (depending on the accuracy used in determining the elements of *Seneca's* orbit)

$x = 0.0016$, thus $M_{Jup} = 3.2\times10^{27}$ kg **(4 points)**

Question 3

(1) Using the *virial theorem* for an isolated, spherical system, i.e, that $-2<K>=<U>$, where "K" is the average kinetic energy and "U" is the average potential energy of the system, determine an expression for the total mass of a cluster of galaxies if we know the radial velocity dispersion, σ, of the cluster's galaxy members and the cluster's radius, R. Assume that the cluster is isolated, spherical, has a homogeneous density and that it consists of galaxies of equal mass.
(2) Find the *virial mass*, i.e. the mass calculated from the *virial theorem*, of the Coma cluster, which lies at a distance of 90 Mpc from us, if you know that the radial velocity dispersion of its member galaxies is $\sigma_{\upsilon_r} = 1000 km/s$ and that its angular diameter (on the sky) is about 4^o.
(3) From observations, the total luminosity of the galaxies comprising the cluster is approximately $L = 5\times10^{12}\, L_\odot$. If the mass to luminosity ratio, M/L, of the cluster is ~1 (assume that all the mass of the cluster is visible mass), this should correspond to a total mass $M \sim 5\times10^{12}\, M_\odot$ for the mass of the cluster. Give the ratio of the luminous mass to the total mass of the cluster you derived in question (2).

Answer:

(1) Using the virial theorem for our isolated, spherical system of N galaxies of mass m, each, we get

$$-2\langle K\rangle = \langle U\rangle \Rightarrow -\frac{2}{N}\sum_1^N \frac{1}{2} m_i u_i^2 = < U > \Rightarrow -\frac{m}{N}\sum_1^N u_i^2 = \frac{U}{N}$$ **(3 point)**

where $\frac{1}{N}\sum_1^N u_i^2 \approx \langle u^2\rangle = \langle u_r{}^2\rangle + \langle u_\theta{}^2\rangle + \langle u_\phi{}^2\rangle$, where u_r, u_θ and u_φ are the radial velocity and the two perpendicular velocities on the plane of the sky of the members of the cluster. **(6 points)**

Assuming that $\langle u_r^{\,2}\rangle \sim \langle u_\theta^{\,2}\rangle \sim \langle u_\phi^{\,2}\rangle$, we have

$$-3m\langle u_r^2\rangle = U/N = (-\frac{3}{5}\frac{GM^2}{R})/N = -\frac{3}{5}\frac{GMm}{R} \Rightarrow M \approx \frac{5R\langle u_r^2\rangle}{G}$$ **(13 points)**

(where we used the gravitational potential of a spherical homogeneous mass M enclosed within radius R)

Alternatively the student can give a rougher order of magnitude estimate

$-2\langle K\rangle = \langle U\rangle \Rightarrow -2.\frac{1}{2}\langle u_r^2\rangle \approx -\frac{GM}{R}$ etc If they do not use the exact formula: (11 Points)

(2) From the result of the previous question we have $M \approx \frac{5R\langle u_r^2\rangle}{G}$, where

$$\sqrt{\langle u_r^2\rangle} = \sigma = 1000\ km\,s^{-1}$$ **(8 points)**

The angular diameter of the cluster is φ = 4° at a distance of *d* = 90 Mpc. Therefore the diameter D of the cluster in Mpc is calculated from:

$$\tan\phi = \frac{D}{d} \Rightarrow \phi(rad) \approx \frac{D}{d} \Rightarrow D \approx 4\times\frac{\pi}{180}\times 90\,\mathrm{M}pc \approx 6.3Mpc \Rightarrow R \approx 3Mpc$$ **(11 points)**

Therefore $M \approx \frac{5R\langle u_r^2\rangle}{G} = 3.6\times10^{15}\mathrm{M}_\odot$ **(3 point)**

(3) $\frac{M_{virial}}{L_{galaxies}} = \frac{3.6\times10^{15}\mathrm{M}_\odot}{5\times10^{12}L_\odot} = 720\frac{\mathrm{M}_\odot}{L_\odot}$ This is obviously much larger from $\frac{\mathrm{M}_\odot}{L_\odot} = 1$ which is found from the visible mass of the cluster. Thus $\frac{M}{M_{virial}} = \frac{1}{720}$. **(6 points)**