Long Problem

Note: 30 points for each problem

16) A spacecraft is launched from the Earth and it is quickly accelerated to its maximum velocity in the direction of the heliocentric orbit of the Earth, such that its orbit is a parabola with the Sun at its focus point, and grazes the Earth orbit. Take the orbit of the Earth and Mars as circles on the same plane, with radius of r_E=1AU and r_M=1.5AU, respectively. Make the following approximation: during most of the flight only the gravity from the Sun needs to be considered.

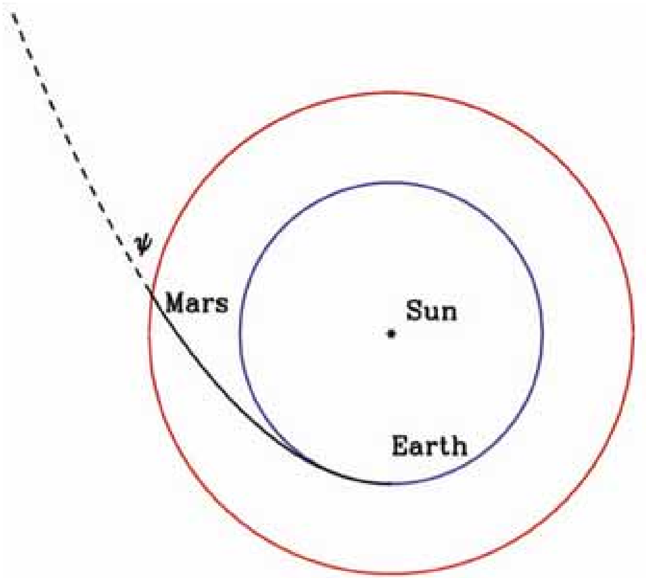


Figure 1:

The trajectory of the spacecraft (not in scale). The inner circle is the orbit of the Earth, the outer circle is the orbit of Mars.

Questions:

(a) What is the angle ψ between the path of the spacecraft and the orbit of the Mars (see Fig. 1) as it crosses the orbit of the Mars, without considering the gravity effect of the Mars?

(b) Suppose the Mars happens to be very close to the crossing point at the time of the crossing, from the point of view of an observer on Mars, what is the approaching velocity and direction of approach (with respect to the Sun) of the spacecraft before it is significantly affected by the gravity of the Mars?

Solution: (1) 10 points; (2) 20 points

(1) The orbit of the spacecraft is a parabola, this suggests that the (specific) energy with respect to the Sun is initially

$$\varepsilon = 1/2v_{\max}^2 + U(r_E) = 0$$ 2

and

$$v_{\max} = \sqrt{2U} = \sqrt{2k_{sun}/r_E}$$

The angular momentum is

$$l = r_E v_{\max} = \sqrt{2k_{sun} r_E}$$ 2

When the spacecraft cross the orbit of the Mars at 1.5 AU, its total velocity is

$$v = \sqrt{2U} = \sqrt{2k_{sun} r_M} = \sqrt{\frac{2}{3}} v_{\max}$$

This velocity can be decomposed into v_r and v_θ, using angular momentum decomposition,

$$r_M v_\theta = l = r_E v_{\max}$$ 2

So,

$$v_\theta = \frac{r_E}{r_M} v_{\max} = \frac{2}{3} v_{\max}$$ 2

Thus the angle is given by

$$\cos\psi = \frac{v_\theta}{v} = \sqrt{\frac{r_E}{r_M}} = \sqrt{\frac{2}{3}}$$

or

$$\psi = 35.26^\circ \qquad 2$$

Note: students can arrived at the final answer with conservation of angular momentum and energy, full mark.

(2) The Mars would be moving on the circular orbit with a velocity

$$v_M \equiv \sqrt{\frac{k_{sun}}{r_M}} = \sqrt{\frac{2}{3}} v_E = 24.32 km/s \qquad 3$$

from the point of view of an observer on Mars, the approaching spacecraft has a velocity of

$$\overrightarrow{v_{rel}} = \vec{v} - \vec{v}_M \qquad 2$$

Now

$$\vec{v} = v \sin\psi \,\hat{r} + v_\theta \,\hat{\theta} \qquad 2$$

with

$$\sin\psi = \sqrt{1-\cos^2\psi} = \frac{1}{\sqrt{3}}$$

So

$$\overrightarrow{v_{rel}} = v\sin\psi\,\hat{r} + (v_\theta - v_M)\,\hat{\theta}$$
$$= \frac{1}{\sqrt{3}}\sqrt{\frac{2k_{sun}}{r_M}}\,\hat{r} + \left(\frac{2}{3}\sqrt{\frac{2k_{sun}}{r_E}} - \sqrt{\frac{k_{sun}}{r_M}}\right)\hat{\theta}$$
$$= \sqrt{\frac{2k_{sun}}{3r_M}}\,\hat{r} + \left(\frac{2}{\sqrt{3}} - 1\right)\sqrt{\frac{k_{sun}}{r_M}}\,\hat{\theta} \qquad 8$$
$$= \sqrt{\frac{k_{sun}}{r_M}}(0.8165\,\hat{r} + 0.1547\,\hat{\theta})$$

The angle between the approaching spacecraft and Sun seen from Mars is:

$$\tan\theta = \frac{0.1547}{0.8165} = 0.1894 \qquad 3$$
$$\theta = 10.72^\circ$$

The approaching velocity is thus

$$v_{rel} = \sqrt{\frac{2}{3} + \left(\frac{2}{\sqrt{3}} - 1\right)^2}\sqrt{\frac{k_{sun}}{r_M}} = 20.21 km/s \qquad 2$$

17) The planet Taris is the home of the Korribian civilization. The Korribian species is a highly intelligent alien life form. They speak Korribianese language. The Korribianese-English dictionary is shown in Table 1; read it carefully! Korriban astronomers have been studying the heavens for thousands of years. Their knowledge can be summarized as follows:

★ Taris orbits its host star Sola in a circular orbit, at a distance of 1 Tarislength.

★ Taris orbits Sola in 1 Tarisyear.
★ The inclination of Taris's equator to its orbital plane is 3°.
★ There are exactly 10 Tarisdays in 1 Tarisyear.
★ Taris has two moons, named Endor and Extor. Both have circular orbits.
★ The sidereal orbital period of Endor (around Taris) is exactly 0.2 Tarisdays.
★ The sidereal orbital period of Extor (around Taris) is exactly 1.6 Tarisdays.
★ The distance between Taris and Endor is 1 Endorlength.
★ Corulus, another planet, also orbits Sola in a circular orbit. Corulus has one moon.
★ The distance between Sola and Corulus is 9 Tarislengths.
★ The tarisyear begins when Solaptic longitude of the Sola is zero.

Korribianese	**English Translation**
Corulus	A planet orbiting Sola
Endor	(i) Goddess of the night; (ii) a moon of Taris
Endorlength	The distance between Taris and Endor
Extor	(i) God of peace; (ii) a moon of Taris
Sola	(i) God of life; (ii) the star which Taris and Corulus orbit
Solaptic	Apparent path of Sola and Corulus as viewed from Taris
Taris	A planet orbiting the star Sola, home of the Korribians
Tarisday	The time between successive midnights on the planet Taris
Tarislength	The distance between Sola and Taris
Tarisyear	Time taken by Taris to make one revolution around Sola

Table 1: Korribianese-English dictionary

Questions:

(a) Draw the Sola-system, and indicate all planets and moons.

(b) How often does Taris rotate around its axis during one Tarisyear?

(c) What is the distance between Taris and Extor, in Endorlengths?

(d) What is the orbital period of Corulus, in Tarisyears?

(e) What is the distance between Taris and Corulus when Corulus is in opposition?

(f) If at the beginning of a particular tarisyear, Corulus and taris were in opposition, what would be Solaptic longitude (as observed from Taris) of Corulus n tarisdays from the start of that year?

(g) What would be the area of the triangle formed by Sola, Taris and Corulus exactly one tarisday after the opposition?

(a) 5 points

(b) 5 points

(c) 3 points

(d) 2 points

(e) 5 points

(f) 5 points

(g) 5 points

Solution: (a) Drawing scaled diagram is impossible. Rough sketch is accepted.

(b) There are 10 days and nights per taris year. The obliquity is 3^o, which means that the planet's rotation is in the same direction as its orbit. Thus, total number of rotations per year is $10 + 1 = 11$.

Note: The obliquity is positive (similar to the Earth / Mars / Jupiter). This means, we have ADD one rotation. Subtracting one rotation by assuming opposite rotation (like the Venus) is incorrect.

(c) By Kepler's third law, $\frac{T^2}{R^3} =$ Constant

$$\frac{T_{en}^2}{R_{en}^3} = \frac{T_{ex}^2}{R_{ex}^3} \quad (1)$$

$$R_{ex}^3 = \frac{1.6^2 R_{en}^3}{0.2^2} \quad (2)$$

$$R_{ex} = \sqrt[3]{64} \quad \text{endorlengths} \quad (3)$$

$$= 4 \quad \text{endorlengths} \quad (4)$$

(d) Using same logic as above

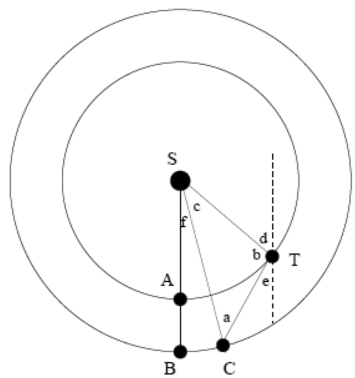
$$\frac{T_C^2}{R_C^3} = \frac{T_T^2}{R_T^3} \quad (5)$$

$$T_C^2 = \frac{9^3 R_T^3 T_T^2}{R_T^3} \quad (6)$$

$$
\begin{aligned}
T_C &= \sqrt{729} \quad \text{tarisyears} && (7)\\
&= 27 \quad \text{tarisyears} && (8)
\end{aligned}
$$

(e) As Corulus is in Opposition, Sola - Taris - Corulus form straight line (in that order).
Distance = 9 - 1 = 8 tarislengths.

(f) In the figure, S is Sola, A and B are start of the year positions of Taris and Corulus, T and C are their positions after 'n' days. Angles are named from a to f. The dashed line is parallel to line SB. Triangle(SCT) is used for sine rule as well as answer in the next part. Figure is not to the scale.



$$
\begin{aligned}
a+b+c &= \pi && (9)\\
b+d+e &= \pi && (10)\\
d &= f+c && (11)\\
f+c &= \frac{2\pi n}{10} && (12)\\
f &= \frac{2\pi n}{270} && (13)\\
\sin b &= 9\sin a \text{ (By Sine Rule)} && (14)
\end{aligned}
$$

$$
\begin{aligned}
e &= \pi - b - d && (15)\\
&= \pi - b - c - f && (16)\\
&= a - f && (17)
\end{aligned}
$$

$$
\begin{aligned}
b &= \pi - (a+c) && (18)\\
&= \pi - \left(a + \frac{2\pi n}{10} - \frac{2\pi n}{270}\right) && (19)\\
&= \pi - \left(a + \frac{52\pi n}{270}\right) && (20)
\end{aligned}
$$

$$
\begin{aligned}
9\sin a &= \sin\left(\pi-\left(a+\frac{52\pi n}{270}\right)\right) && (21)\\
&= \sin\left(a+\frac{52\pi n}{270}\right) && (22)\\
&= \left[\sin a\cos\left(\frac{52\pi n}{270}\right)+\cos a\sin\left(\frac{52\pi n}{270}\right)\right] && (23)\\
9 &= \cos\left(\frac{52\pi n}{270}\right)+\cot a\sin\left(\frac{52\pi n}{270}\right) && (24)\\
\cot a &= \frac{9-\cos\left(\frac{52\pi n}{270}\right)}{\sin\left(\frac{52\pi n}{270}\right)} && (25)\\
a &= \tan^{-1}\left[\frac{\sin\left(\frac{52\pi n}{270}\right)}{9-\cos\left(\frac{52\pi n}{270}\right)}\right] && (26)
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= \pi-e && (27)\\
&= \pi+f-a && (28)\\
\lambda &= \pi+\frac{2\pi n}{270}-\tan^{-1}\left[\frac{\sin\left(\frac{52\pi n}{270}\right)}{9-\cos\left(\frac{52\pi n}{270}\right)}\right] && (29)
\end{aligned}
$$

(g)

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}\times l(ST)\times l(SC)\times\sin\mathbf{c}\\
&= \frac{1}{2}\times 1\times 9\times 0.568\\
&= 2.56
\end{aligned}
$$

The area is about 3(tarislength)²