Theoretical round. Basic criteria. For work of Jury

Note. The given sketches are not full; the team leaders have to give more detailed explanations to students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

Note. Jury members should evaluate the student's solutions in essence, and not by looking on formal existence the mentioned sentences or formulae. The formal presence of the mentioned positions in the text is not necessary to give the respective points. Points should be done if the following steps de facto using these positions.

α-1. Satellite of Mars.

Correct distance from Mars, $A \approx Lr/R \approx 1.1$ million.km – 1 pt.
Idea to compare this distance with the size of the Hill sphere – 1 pt.
Distance to the positions of the Lagrange points – 2 pt.
Conclusion $A > \Lambda$ – 1 pt.
Conclusion «**the situation is impossible**» – 1 pt.
A figure that is needed to accompany the solution – 2 pt.
(If wrong issue about possibility) Calculation of a hypothetical period of the satellite – 1 pt.

β-1. Dyson sphere.

Initial size of the star does not matter (period depends on the mass) – 1 pt
Period depends on the density of the modern Betelgeuse – 1 pt.
Using difference in luminosities – 2 pt.
Using irradiation law – 1 pt.
Finding the density of the modern Betelgeuse – 1 pt.
Finding orbital period – 1 pt.
Correct accuracy in the answer (not more 2 significant digits) – 1 pt.

α-2. Length of Day.

Understanding to use law of conservation of angular momentum – 1 pt.
Correct equation of conservation of angular momentum – 1 pt.
Explanation what we can neglect – 1 pt.
Correct using all necessary parameters – 1 pt.
Algebraic transformations and correct formula for $\Delta h/\Delta T$ – 2 pt.
Final result – 2 pt.

β-2. Length of Day.

Understanding to use law of conservation of angular momentum – 1 pt.
Correct equation of conservation of angular momentum – 1 pt.
Explanation what we can neglect – 1 pt.
Correct using all necessary parameters – 1 pt.
Algebraic transformations and correct formula for $\Delta h/\Delta T$ – 2 pt.
Subtraction 1.6 ms / century – 1 pt.
Final graph – 1 pt.

αβ-3. Heaven omen. Two comets.

$a = 3^{2/3} = 2.08$ au – 1 pt.
A = 2.8 au (or 2.77) P = 2a – A = 1.36 au – 1 pt.
Conclusion that comets pass near the ecliptic plane both aphelion and perihelion – 1 pt.
Comets 0.5 revolutions = Earth 1.5 revolutions – 1 pt.
Kepler's Second Law for the positions of aphelion and perihelion – 2 pt.
Formulae for the angular distances as seen from the Earth – 1 pt.
Correct result – 1 pt.

αβ-4. Heaven omen. Moon and comet.

4.1. 2 pt, including:
The comet goes from under the right side of the Moon – 2 pt.
Evening – 1 pt.
4.2. 2 pt, including:
Position of the Sun and the Moon relative to the Sun – 2 pt.
Gemini – 1 pt.
4.3. The Moon in the west or north-west and any answer based on this fact – 1 pt.
4.4. Artistic drawing – 1 pt.
4.5. 2 pt for the total full solution and dates of June 7-9, not more 1 pt for partial solution.

αβ-5. Search for asteroids.

Using distance from the Sun – 1 pt.
Using distance from the Earth – 1 pt.
Using sizes of the bodies – 1 pt.
Using sizes of albedo – 1 pt.
Formula using all the previous parameters and algebraic transformations – 2 pt.
Final result – 1 pt.
Correct accuracy in the answer (not more 1 significant digit) – 1 pt.

Theoretical round. Sketches for solutions. For jury job ONLY

Note. The given sketches are not full; the team leaders have to give more detailed explanations to students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

Note. Jury members should evaluate the student's solutions in essence, and not by looking on formal existence the mentioned sentences or formulae. The formal presence of the mentioned positions in the text is not necessary to give the respective points. Points should be done if the following steps de facto using these positions.

α-1. Satellite of Mars. At first, let us find the distance A, at which the satellite should be to see from an eclipse of the Sun by Mars as well "as how on Earth". Let R be the radius of the Sun, r is the radius of Mars, L is the distance from the Sun to Mars. From the equality of angular dimensions:

$$r/A = R/(L+A), \quad A = L\cdot r/(R-r) \approx Lr/R \quad \text{(as } r << R).$$

For the average distance from the Sun to Mars 1,524 au,

$$A = 1.524\cdot 149.6 \text{ million.km.} \times 6794 \text{ km} / 1392000 \text{ km} \approx 1.11 \text{ million.km.}$$

Next, the main idea of the problem lies in the fact that to compare this distance with the size of the Hill sphere, in other words, the distance from Mars to the first Lagrange point. Calculating the position of the Lagrange points in general is difficult, but in the case of Mars and the Sun, we can use the approximation $M_M << M_S$.

$$GM_S/(R+\Lambda)^2 + GM_M/\Lambda^2 = \omega^2(R+\Lambda),$$

in this case, the condition of Mars in its orbit $GM_S/R^2 = \omega^2 R$ we obtain $\omega^2 = GM_S/R^3$,

$$M_S/(R+\Lambda)^2 + M_M/\Lambda^2 = M_S(R+\Lambda)/R^3.$$

Solving this equation using the approximation $M_M << M_S$, $\Lambda << R$, we see that the second Lagrange point is located at a distance from Mars

$$\Lambda = (M_M/3M_S)^{1/3} = 1.08 \text{ million.km.}$$

Thus, $$A > \Lambda.$$

The orbit of the satellite was supposed to be outside the Hill sphere. Mars will inevitably lose a satellite.

A figure that is needed to accompany the solution should describe the needed position of the satellite for eclipse observations and Hill sphere or Lagrange points.

Answer: «**the situation is impossible**».

However, you can find a hypothetical period of the satellite in orbit with radius A around Mars.

$$T = 2\pi(A^3/GM_M)^{1/2} = 3.6\cdot 10^7 \text{ s} \approx 415 \text{ days.}$$

It is 415/687 = 0.6 orbital period of Mars, which is more than a critical factor $1/3^{1/2}$: there are no orbits around the planets, which period of revolution exceeds $1/3^{1/2}$ periods of revolution of the planet around the star.

β-1. Dyson sphere. As is known, the orbital period of a planet in a circular orbit around a star depends on the mass of the star only. Therefore, the initial size uf the star does not matter. Let us derive a formula for this period in the extreme case, when the radius of the orbit is equal to the radius of the star R:

$$\tau = 2\pi / \omega, \quad \omega^2 \cdot R = GM/R^2, \quad \omega^2 = GM/R^3 = 4/3\ \pi\ G\rho, \quad \tau = (3\pi/G\rho)^{1/2}.$$

Thus, the period depends only on the density of the ball, directly along the surface of which this circular movement occurs, and in our case – on the density of the current Betelgeuse.

All necessary data may be taken from tables.

The magnitude of Betelgeuse is $0^m.5$, and the distance to it is

$$D_B = 206265\ \text{au/pc} \times 197\ \text{pc} \approx 40\ 600\ 000\ \text{au}.$$

Our Sun being remote to this distance has the magnitude

$$\mathbf{m_1} = -26^m.74 + 5^m \lg 40\ 600\ 000 \approx -26^m.74 + 38^m.04 \approx 11^m.30.$$

So the difference in absolute stellar magnitudes between Sun and Betelgeuse is

$$\Delta\mathbf{m} = 11^m.3 - 0^m.5 = 10^m.8.$$

Thus, the ratio of luminosities of Betelgeuse L_A and the Sun L_0 is

$$L_B / L_0 = 100^{10.8/5} \approx 21\ 000\ \text{times}.$$

The luminosity of a star L is proportional to its surface area and the fourth power of surface temperature, i.e. $L \sim R^2T^4$. The density of a star is equal to its mass divided by its volume, i.e. it is proportional to M/R^3. Thus, $\rho \sim MT^6/L^{3/2}$. The ratio of masses of Betelgeuse and Sun, their temperatures may be found from the table of stars. They are 11.6, 3590 K and 5777 K respectively. So by comparing the densities of Betelgeuse and Sun (see table of Solar System, $\rho_0 \approx 1410$ kg/m^3), we may find that the density of Betelgeuse is

$$\rho_B = \rho_0 \cdot (M_B/M_0) \cdot (T_B/T_0)^6 \cdot (L_B/L_0)^{-3/2} \approx 3.1 \cdot 10^{-4}\ \text{kg/m}^3.$$

In our case

$$\tau = (3\pi/G)^{1/2} \cdot \rho_0^{-1/2} \cdot (M_B/M_0)^{-1/2} \cdot (T_B/T_0)^{-3} \cdot (L_B/L_0)^{3/4} \approx 2.13 \cdot 10^7\ \text{s} \approx 247\ \text{days}.$$

However, the problem is only for estimation, therefore, an accuracy of more than two significant digits in the answer is inappropriate, and the correct answer is: **about 250 days**.

α-2. Length of Day. If the only reason of the "subdecadal" variations variations is fluctuations in sea level, the physical reason for this is the law of conservation of angular momentum.

$$L = \text{const}.$$

Let us start to solve the problem qualitively. Given that the angular momentum is the product of the momentum of inertia of the system by the angular velocity $L = I\omega$, and the moment of inertia is larger if the system is "wider" (more weight is farther from the axis of rotation), it can be concluded that an increase in global sea level (due to the movement of melting ice from the polar regions to the entire surface of the ocean) leads to an increasing in the moment of inertia of the Earth. Accordingly, it reduces the rotation speed of the Earth, that is, the rotation period is increasing. At our figure the situation is opposite, from year 1995 to 2003, the length of the day decreased by 1.8 ms (on the contrary, the water from the ocean surface has settled near the poles), respectively, and we can immediately conclude that $\Delta h = -$...

Our problem is to estimate, so from the beginning we neglect the difference between the duration of the day and the period of revolution around own axis and the difference of the shape of Earth from the sold ball.

Let M_0 be the mass of the Earth without the circumpolar ice layer, which will be melt later. R is the average radius of the Earth, and ω is the angular speed of rotation of the Earth:

$$L_0 = I_0 \cdot \omega_0 = (2/5\ M_0 R^2 + I_{ice}) \cdot \omega_0$$

After a while some of the ice melts:

$$L_1 = I_1 \cdot \omega_1 = (2/5\ M_0 R^2 + 2/3\ \eta\ \rho\ S\ \Delta h\ (R+\Delta h/2)^2) \cdot \omega_1,$$

where S is the surface area of the Earth, ρ is the density of water, η is the ratio of ocean area surface to the total surface of the globe, so η ρ S Δh is the mass of water that raise the level of the ocean to the Δh.

$$L_1 \approx (2/5\ M_0 R^2 + 2\eta/3\ \rho\ 4\pi R^2 \Delta h R^2) \cdot \omega_1 \approx (2/5\ M_0 R^2 + 8\pi\eta/3\ \rho R^4 \Delta h) \cdot \omega_1,$$

Since $L_0 = L_1$, $I_0 \cdot \omega_0 = I_1 \cdot \omega_1$,

$$(2/5\ M_0 R^2 + I_{ice}) \cdot \omega_0 = (2/5\ M_0 R^2 + 8\pi\eta/3\ \rho R^4 \Delta h) \cdot \omega_1,$$

Since the polar ice is concentrated near the poles, the value I_{ice} is negligible, and we also neglect it.

$$8\pi\eta/3\ \rho R^4 \Delta h \cdot 2\pi/T_1 = 2/5\ M_0 R^2\ (2\pi/T_0 - 2\pi/T_1) = 2/5\ M_0 R^2\ 2\pi(T_1 - T_0)/T_1\ T_0$$

and since $T_1 \approx T_0 \approx T$,

$$8\pi\eta/3\ \rho R^4 \Delta h = 2/5\ M_0 R^2\ \Delta T/T.$$

$$\Delta h/\Delta T = 3/20\pi\eta\ M_0/T\rho R^2.$$

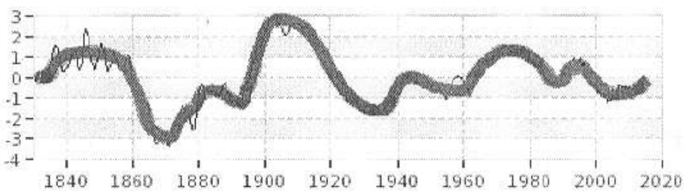
Taking the necessary data from the tables, taking η = 0,71 (values from 2/3 to 3/4 are considered correct) and taking the salt water density ρ = 1024 kg/m^3, we obtain the relation between variations in the global sea level and the duration of the day (without taking into account the influence of the Moon):

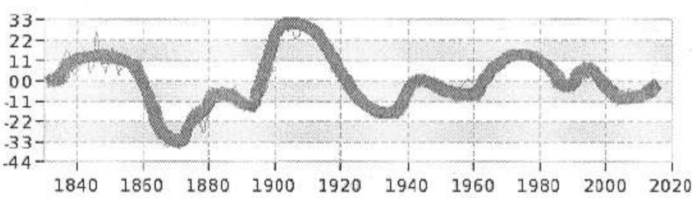
$$\Delta h/\Delta T = 0.11\ m/ms.$$

From year 1995 to 2003 the duration of the day decreased by 1.8 ms, so

$$\Delta h = 0,11\ m/ms \times -1,8\ ms \approx -0,2\ m = -20\ cm.$$

(The Polar ice, on the contrary, the ice grew and did not melt.)





β-2. Length of Day. First part of the solution – see solution for the group Alfa.

In order to use of the above-obtained ratio Δh/ΔT = 0.11 m/ms, it is necessary to separate the "subdecadal" variations from the "lunar trend", it is necessary to subtract from this graph a a graph of straight line with a slope of 1.6 ms / century. Because we need to draw an indicative plot, short-term fluctuations rounded. The result is a graph shown to the right top. Now we only need to redraw the graph by changing milliseconds along the y-axis by centimeters with a factor 11 cm/ms. The result is about the figure to the right bottom (centimeters are along the y-axis).

αβ-3. Heaven omen. Two comets. According to Kepler's III law we can calculate the semi-major axis of the orbit of the comets. If we measure the period in years and the distance in astronomical units, then for the orbits around the Sun, the law is written as:

$$T^2 / a^3 = 1.$$

In our case T = 3 years, $a = 3^{2/3} = 2.08$ au. We assume that the middle of the asteroid belt, which the comets reach at aphelion is a circle with a radius equal to the mean distance of Ceres from the Sun, A = 2.8 au. Then at perihelion they are at distance P = 2a – A = 1.36 au from the Sun. Since the plane of the asteroid belt is approximately the ecliptic plane, it means that the comets pass near the ecliptic plane both aphelion and perihelion. The orbital period of the comets is 3 years, so while at aphelion they are in opposition, they will also be in opposition at perihelion, having made 0.5 revolutions around the Sun, while the Earth will make 1.5 revolutions.

That the comets are moving along the same path means, that one of them exactly repeats the positions of the other through some constant period of time. Accordingly, the distance L between them is proportional to their speed. According to Kepler's Second Law for the positions of aphelion and perihelion $V_A \cdot A = V_P \cdot P$. In this way

$$L_A \cdot A = L_P \cdot P.$$

The angular distances as seen from the Earth:

$$\beta = L_A / (A-1) \text{ (aphelion)},$$

$$\beta = L_P / (P-1) \text{ (perihelion)}.$$

When viewed in the aphelion to the naked eye the comets merge into one visible point, then the angular distance between them is not more than the resolution of the eye. We assume that the resolution of the eye is equal to δ = 1'. Thus, α < δ,

$$\beta = L_P / (P-1) = L_A \cdot (A/P) / (P-1) = \alpha \cdot (A/P) \cdot (A-1) / (P-1) \approx 10{,}3\,\alpha < 10{,}3\,\delta \approx 10'.$$

Thus, the two comets may well be at an angular distance of one third of the Moon diameter and so impress the warriors.

αβ-4. Heaven omen. Moon and comet. Adrianople is located in the northern hemisphere, all the arguments and directions are given for the northern hemisphere.

4.1. The Moon in its orbit moves from right to left with respect to the stars and planets. So, the comet can go from under the illuminated part of the Moon only when its right side is illuminated, that is, with the rising Moon, and thus the period of day is evening.

4.2. If we are talking about a thin crescent Moon, comparable in brightness with a comet, the age of the moon is maximum 2.5 days. So the Moon is to the left from the Sun at a distance of no more than 30°. During the 15-day opposite standing from 7 to 22 June, the Sun is in the constellation of Taurus, and the Moon, respectively is in Gemini constellation.

4.3. The omen was at sunset. The Moon in this case was in the west or north-west, as it happens in the middle of summer. Thus, the Byzantine army saw this phenomenon in front, and the Bulgarian army from behind. Further arguments may be different. In the annals of the situation described most often, when the army, seeing ahead of bad character unfolded retreated. By this logic, Greeks should have been frightened. But someone might suggest that Bulgarian soldiers were more scared, as it is their "knife in the back".

4.4. Artistic drawing.

4.5. As a reference point we can take evening October 3, 2016 in Pamporovo, when a very thin crescent of the waxing Moon was visible in the sky. Let us count how many days have passed since June 22, 813 A.D. till October 3, 2016.

$$N = (2016–813)*365 +]2016\text{-}813[/4 – 9 +103 = 439490.$$

How many lunar months have passed since then? For this it is necessary to divide this number of days by the synodic period of the Moon. We remember that it is approximately equal to 29.5 days, a little more precise value is 29.53 days.

$$439490 / 29.53 = 14882.83,$$

and by using the fractional part of this number, we can find the difference between the phases of the Moon and get that the same phase was on 17 June 813. However, it is a blunder. If we take the synodic period of the Moon equal to 29.53 days, the error is 0.005 days per lunar month. Multiplying the number of elapsed months by this error, we obtain the total error 14883 × 0.005 = 74 days. Thus, the crescent was seen on June 16, plus or minus 74 days.

To calculate the date with an accuracy of at least one day, we have to reduce the error of the synodical period of at least in two orders of magnitude, i.e., to use value with at least two more significant digits.

It's possible. We are provided with the orbital period of the Moon with six significant digits and the length of the sidereal year with eight significant digits. By using these data we can find synodic period of the Moon with six significant digits:

$$T = T_2T_1/(T_2–T_1) = 365.25636 \times 27.3217 / (365.25636 – 27.3217) = 29.5306.$$

Given this accuracy

$$439490 / 29.5306 = 14882.47,$$

and the fractional part of this number shows us that on June 22 the Moon phase was close to the full moon, while the described crescent could be seen 13.9 days (plus or minus ¾ days) until June 22. That is, it turns out that the phenomenon has been observed in the sky on June 7-9, that is, in the beginning of standing.

αβ-5. Search for asteroids. The same intensity of light should be registered by the telescope form the asteroid of the main asteroid belt with size D = 2.5 km and the yet unknown asteroid of the Kuiper belt. The following aspects should be taken into account:

1. **R**. Distance from the Sun: 2.8 au for the asteroid of the main belt and about 35-50 au for the asteroid of the Kuiper belt. The intensity of light coming to the asteroids is reverse proportional to the square of these distances.
2. **L**. Distance from the observer (i.e. from the Earth): 1.8 au for the asteroid of the main belt (in oppositions) and about 35-50 au for asteroid of the Kuiper belt. The intensity of light coming to the telescope is reverse proportional to the square of these distances.
3. **D**. Size (diameter) of the body. The intensity of light reflecting by the body is proportional to the square of this size.
4. **α**. Albedo. Let us consider the physical composition of the objects to be similar to Ceres and Makemake respectively (Eris is outside the Kuiper belt). The albedo of Eris is about 0.09

and the one of Makemake is 0.77. The intensity of light reflecting by the body is proportional to the albedos.

So $I \sim R^{-2} \cdot L^{-2} \cdot D^{2} \cdot \alpha$. And for $I_1 = I_2$ we have to write: $D_2/D_1 = (R_2/R_1) \cdot (L_2/L_1) \cdot (\alpha_1/\alpha_2)^{1/2}$

$$D_2 = 2.5 \text{ km} \cdot 15 \cdot 23 \cdot (0{,}12)^{1/2} \approx 290 \text{ km}.$$

The problem is to estimate, so more accuracy with two significant digit is not appropriate

Answer: about **300 km**.