2016 International Astronomy Olympiad

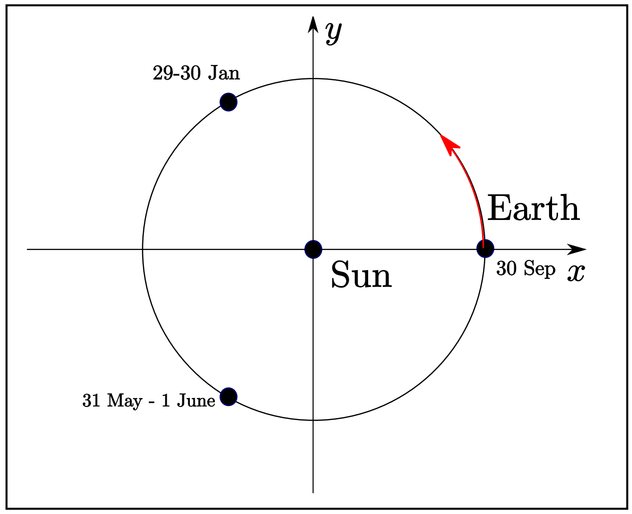
Practical round: solutions

The gravitational constant is γ or G = 6.67x10^{-11} m^3 kg^{-1} s^{-2}

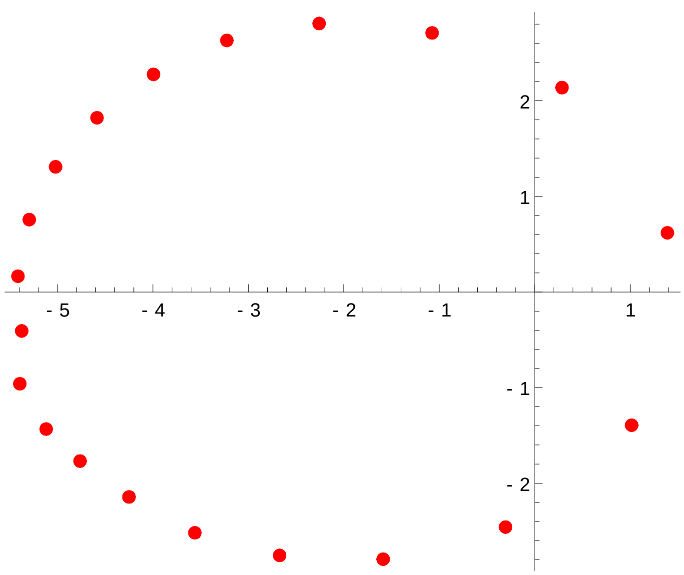
Problem α-6. Comet observer.

6.1. *2pts*

We start off by drawing the Earth's orbit. Using the tabular data we find that the suitable value for the radius is 2 cm. Because there are 3 observations per year, as noted in the text, there are just 3 positions of the Earth, where the observations take place: one for **30 September**, one for **29-30 January** and one for **31 May-1 June**. We mark these positions on the graph paper. Because 30 Sep 2004 is when Delta takes the largest tabular value (6.485 AU), it is convenient to place 30 Sep on the x-axis. The other two observational points are separated by an angle of 120° and are easily marked using simple geometry.



Next we draw all 20 points from the table on the graphing paper, where each point represents the comet's position for a particular observation with given N . To this end, we use only columns **Date(UT)**, **Delta** and **S-O-T** and disregard the rest. Note that **Delta** is the distance between the comet and the Earth. Therefore some care must be exercised to measure **Delta** from the correct position of the Earth defined by **Date(UT)**.



Next we draw a smooth closed curve around the points, which resembles an ellipse.

6.2. *3pts*

First by observing the curve we identify the direction of the semi-major axis. Then we draw a line along this direction, which approximately halves the ellipse. Next we mark the so-obtained perihelion and aphelion and measure the distance between them, which yields $2a$. We also measure the perihelion distance, which yields $a(1-e)$. Bearing in mind that 2 cm = 1 AU, we find that

$$a \approx 3.45 \text{ AU and } e \approx 0.58.$$

In fact, the real values are $a \approx 3.44$ AU and $e \approx 0.57$.

6.3. *1pt*

The first point ($N=1$)and the last point ($N=20$) almost lie on the x-axis meaning that for the time interval separating these observations, the comet has done a full cycle. Therefore we obtain

$$T \approx (20-1)\frac{1}{3}\text{years} \approx 6.33\,\text{years},$$

while the real value is $T \approx 6.38$ years.

6.4. *2pts*

From Kepler’s second law we know that a line segment joining an orbitingbody and the Sun sweeps out equal areas ΔS during equal intervals of time Δt. This means that $\Delta S/\Delta t = S/T$, where S isthe area inside the ellipse, and T is the comet’s orbital period. Note that at the perihelion and at the aphelion the following identities hold for very small Δt:

$$\frac{\Delta S}{\Delta t} = \frac{1}{2}v_p a(1-e) \text{ and } \frac{\Delta S}{\Delta t} = \frac{1}{2}v_a a(1+e).$$

For the derivation consider the area of an isosceles triangle of height $a(1-e)$ and base $v_p \Delta t$.

Thus we find

$$v_a = \frac{2S/T}{a(1-e)} \approx 31.5 \text{ km/s and } v_p = \frac{2S/T}{a(1+e)} \approx 8.4 \text{ km/s}.$$

We can express the speeds directly by a and T if we use the formula $S = \pi ab = \pi a^2\sqrt{1-e^2}$:

$$v_a = \frac{2\pi a}{T}\sqrt{\frac{1-e}{1+e}} \approx 31.5 \text{ km/s and } v_p = \frac{2\pi a}{T}\sqrt{\frac{1+e}{1-e}} \approx 8.4 \text{ km/s.}$$

6.5. *2pts*

The solar mass M can be obtained directly from the identity (neglecting the comet's mass): $T^2 = \frac{4\pi^2 a^3}{\gamma M}$. Thus we have

$$M = \frac{4\pi^2 a^3}{\gamma T^2} \approx 2\text{x}10^{30} \text{ kg.}$$

6.6. *2pts*

From conservation of energy we have $\frac{mv^2}{2} - \frac{\gamma Mm}{r} = \frac{mv_p^2}{2} - \frac{\gamma Mm}{a(1-e)}$, which gives

$$v^2 = v_p^2 + 2\gamma M\left(\frac{1}{r} - \frac{1}{a(1-e)}\right).$$

In solution 6.4. we found $v_p = \frac{2\pi a}{T}\sqrt{\frac{1+e}{1-e}}$. From $T^2 = \frac{4\pi^2 a^3}{\gamma M}$ we obtain $v_p^2 = \frac{\gamma M}{a}\frac{1+e}{1-e}$.

Thus we arrive at

$$v = \sqrt{\gamma M\left(\frac{2}{r} - \frac{1}{a}\right)}.$$

For the observation with $N = 7$ we measure that $r = 3.25$ AU, so we get $v_7 \approx 17.2$ km/s.

The escape velocity *at this point* is obtained by taking the limit as a approaches ∞:

$$v_e = \sqrt{\frac{2\gamma M}{r}} \approx 23.7 \text{ km/s.}$$

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Problem β-6. The first gravitational wave detection.

6.1. *1 pt*

The initial masses of the two black holes are 36+/-4$M_\odot$ and 28+/-4$M_\odot$. Their Schwarzschild radii are calculated using $R_s = 2GM/c^2$, which gives 107+/-12 km and 83+/-12 km. The errors are calculated using ($\Delta R_s/R_s = \Delta M/M$).

6.2. *2.5 pts*

A good estimate of the precision of the measurements can be gained by comparing the Hanford and Livingston signals. At the time of the merger, we see two strong maxima and a minimum of the signal. For them, the time-axis intercepts for the Livingston signal are approximately at:

[0.4190, 0.4215, 0.4234, 0.4259] s

while for the Hanford signal, they are at:

[0.4193, 0.4217, 0.4239, 0.4259] s

These correspond to half-periods of:

[0.0025, 0.0019, 0.0025] sec
[0.0024, 0.0022, 0.0020] sec

for the two signals, respectively.

Assuming those measurements are both independent and representative for the time of the merger, taking the average and standard deviation of those, we get the half-period of the signal to be:

0.0023±0.0003s

An attempt at a more accurate determination can be made by building a P(t) diagram.

The signal has close to symmetric minima and maxima, and the trends of both the maxima and the minima are consistent across both the odd and even peaks. This symmetry implies that the signal is emitted from a binary system with two components of roughly equal masses, as confirmed by the LIGO results.

Therefore, from the symmetry of the configuration of the system, the signal must be identical irrespective of which of the initial black holes is closer to us while they orbit each other. Thus, the signal must have a period of exactly half of the orbital period of the black holes. Therefore, the orbital period at the time of the merger must be 4 times larger than the half-period of the signal quoted above.

Thus, we can conclude that the orbital period is:

$T = 0.0092 \pm 0.0012$ s

6.3. *2 pts*

Assuming circular orbits and Newtonian mechanics, we can apply Kepler's third law:

$a^3/T^2 = G(M_1+M_2)/(4\pi^2)$

where the semi-major axis at the moment of the merger is simply:

$a = R_{s1}+R_{s2} = 2G(M1+M2)/c^2$

We can combine the two equations above to solve for the total mass:

$M_1+ M_2 = Tc^3/(2^{5/2}\pi G)$

Using the value for the period we obtained in II.b, we find:

$M_1+M_2=105\pm13M_\odot$

6.4. *1.5 pts*

In Newtonian mechanics, the gravitational potential energy of the system is

$E_P = -GM_1M_2/a$.

For a circular orbit the kinetic energy is $E_K = |E_P|/2$, and so the mechanical energy is given by

$E = -GM_1M_2/(2a)$,

which at large separations tends to zero. Assuming, the orbits are close to circular, and using a semi-major axis of $a=R_{s1}+R_{s2}$ at the time of the merger, we obtain that the change in mechanical energy of the binary system until the moment of the merger is given by:

$\Delta E = (M_1M_2/(M_1+M_2))(c^2/4)$

which should equal the total energy emitted in gravitational waves. Plugging in the numerical results obtained so far, and the assumption M1=M2, we find:

$\Delta E = 6.6 \pm 0.8 M_{\odot} c^2$.

6.5. *2 pts*

We will find the period around t~0.32s. The two maxima are around that time are at:

0.316±0.005s and 0.344±0.005s

The period of the signal is then 0.028±0.005s (errors added in quadrature). This implies an orbital period which is twice larger (see II.b):

Ti=0.056±0.010s at 0.10 s before the outburst

Combining the expression for the mechanical energy:
$E = -GM_1M_2/(2a)$

with Kepler's third law:

$a^3/T^2=G(M_1+M_2)/(4\pi^2)$,

we can find the mechanical energy in the initial moment. We obtain:

$Ei = 2.0 \pm 0.5 M_{\odot} c^2$

Our previous estimate of E was made in 6.4.:

$E = 6.6 \pm 0.8 M_{\odot} c^2$ at t = 0.4226±0.0002s

Thus, the change of mechanical energy of the system is:
$\Delta E_{0.1} = Ei - E = 4.6 \pm 0.9 M_{\odot} c^2$

over a time interval of 0.1s

This corresponds to average power of emission for the chosen time interval, which equals:

$P = \Delta E_{0.1}/\Delta t = 8 \pm 2 \times 10^{48}$ W

6.6. *2 pts*

We are told that the flux (F) is proportional to h^2. We also know that for any wave, F is inversely proportional to the square of the distance (d). Then h and d must be inversely related ($F \sim h^2$, $F \sim 1/d^2 \rightarrow h \sim 1/d$). Thus, we can set up the following ratio:

$h_1/h_2=d_2/d_1$

for any two points outside the black holes.

Right next to the two black holes (at a distance $R_s \sim R_{s1} + R_{s2}$) from the center of masses), the strain is $h_0 = v^2/c^2$. We can estimate the distance d via

$$h/h_0 = R_s/d \quad \rightarrow \quad d = R_s(v/c)^2/h$$

where h is the strain measured here on Earth. From the figure, $h \sim 1x10^{-21}$.

If we approximate the orbital velocity at the time of merging as circular, if would be

$$v^2 \sim G(M_1+M_2)/(R_{s1}+R_{s2}) \rightarrow (v/c)^2 \sim 0.5$$

Therefore, to an order of magnitude d ~ 1.7 Gpc.

6.7. *1pt*

Supermassive black holes span the range from $10^6 M_\odot$ to $10^{10} M_\odot$. Using a cosmological distance of $d_{min} \sim 1$Gpc to such merging sources, combined with our result for how the strain scales with distance from the previous part of the problem:

$$h \sim 0.5(R_{s1}+R_{s2})/d$$

we get:

$h \sim$ from 10^{-17} to 10^{-13}

for the signal from such merging SMBHs. To be detectable, we need a non-trivial signal-to-noise ratio, and thus we can put the minimum sensitivity requirement at

$h_{min} \sim$ from 10^{-18} to 10^{-14} for the range of masses, corresponding to SMBHs.

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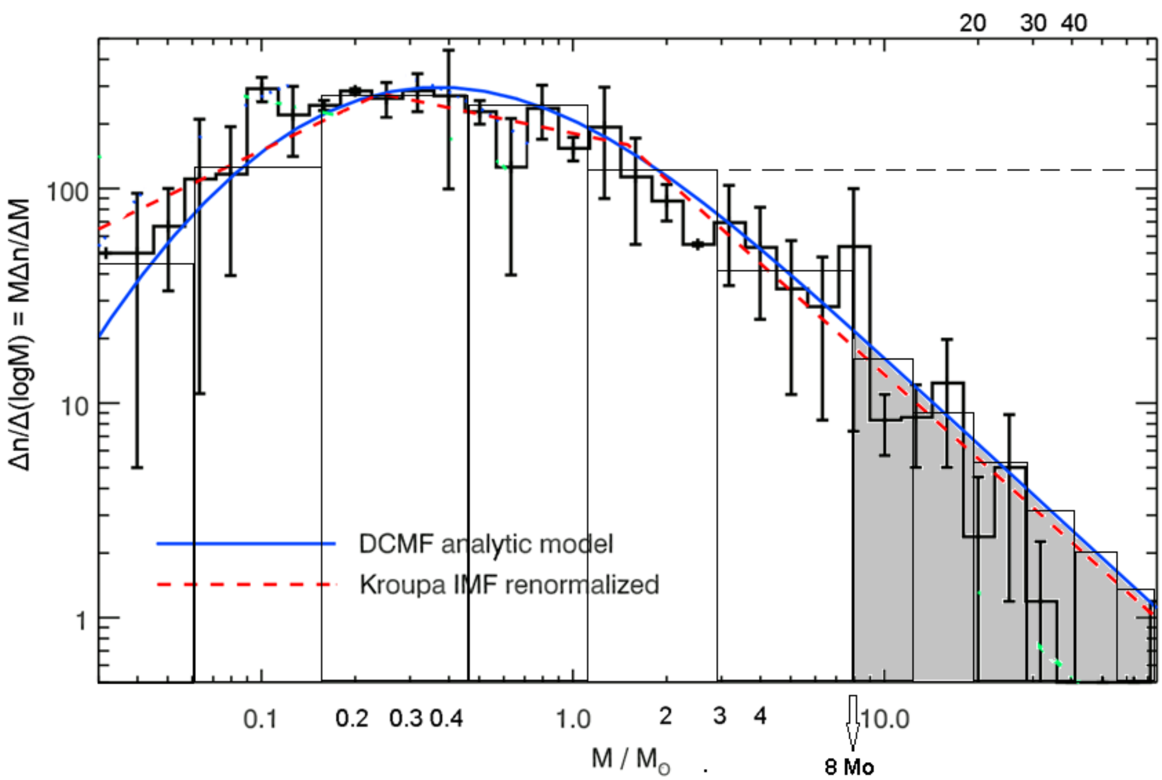
Problem αβ-7. The Initial Mass Function and supernovae.

7.1. *6pts (3pts for reasonably accurate measurements + 3pts for further considerations)*



Stars with an initial mass > 8 $M_\odot$ explode as core-collapse supernovae. Let us mark on the IMF figure the space where these stars reside with grey. Then we divide both the left and the right part of the graph with bins. We calculate the scales on both axes:

X: Δ(lgM) = 1 → ΔX=5.75 cm
Y: Δ(lg(Δn/ΔlogM)) = 1 → ΔY=3.95 cm (defining lgX = $\log_{10}$X)

The base of the Y-axis logarithm is irrelevant – if it is changed, the function is multiplied by a constant and the final results should be the same.

For each bin we calculate the width Δ(lgM) and the height lg(Δn/ΔlogM). We make the rough approximation that Δ(lgM) << 1 so the process is analogous to numerical integration and can be expressed with integrals as well. For further accuracy the function can be extrapolated linearly until it crosses the X-axis at ~ 160 $M_\odot$.

For each bin we then calculate the X-axis center M and the value of Δn that corresponds to the bin width. If the values to the left of the 8 $M_\odot$ vertical line are M_L, Δn_L and to the right are M_H, Δn_H, we can obtain

$$M_{SN} = \sum(M_H \Delta n_H)/\sum(\Delta n_H)$$

$$q = \sum(M_H \Delta n_H)/\sum(M_H \Delta n_H + M_L \Delta n_L)$$

A reasonable result would be $M_{SN} \sim 20\ M_\odot$, q ~ 30%.

Other mathematical methods and bin distributions may be used so these values are correct within an order of magnitude. Participants from group α can avoid the usage of logarithms by considering the Y-axis in the form M(Δn/ΔM) and by doing linear approximations between the X-axis ticks (or by choosing bins using the ticks).

As the galactic star formation rate is said to be ΔM/Δt = 8 $M_\odot$/yr, and the timescale of the evolution of massive stars is much lower than the change of this rate, the mass rate of supernova explosions is

qΔM/Δt ~ 2.4 $M_\odot$/yr

This mass is distributed among objects with an average mass M_{SN} therefore the expected frequency is

$f = q(\Delta M/\Delta t) / M_{SN} \sim 0.12\ yr^{-1}$

or the derived expectations are that there should be a supernova each 8 years on average. Please note that the mass loss of massive stars is not taken into account which leads to significant systematic errors. The mass of supernova progenitors right before the explosion is significantly lower than their initial mass. The actual values are $M_{SN} \sim 12\ M_\odot$, q ~ 8%, and $f = 1/20\ yr^{-1}$.

7.2. *2pts*

The last directly observed galactic supernovae exploded in 1006, 1054, 1572 and 1604 and some of them were type Ia, which are less massive systems with a different mechanism of explosion. So the approximate value of the observed frequency is

$f \sim 1/400\ yr^{-1}$

This value is significantly lower than the one derived in 7.1. A possible explanation is that many explosions are obscured by dust as massive stars are situated in the galactic disk. As we lie inside it, the galactic disk covers a large angular area on the sky which is not easy to monitor. Furthermore, most of the observations historically were conducted from the Northern Hemisphere.