Theoretical round. Sketches for solutions. **For jury job ONLY**

Note. The given sketches are not full; the team leaders have to give more detailed explanations to students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

Note. Jury members should evaluate the student's solutions in essence, and not by looking on formal existence the mentioned sentences or formulae. The formal presence of the mentioned positions in the text is not necessary to give the respective points. Points should be done if the following steps de facto using these positions.

Theoretical round. Basic criteria. For work of Jury

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αβ-1. **Noon at the Olympiad.** From the graph of the Equation of Time we see that now (and until approximately the first days of November) the apparent noon (the culmination of the Sun) every day comes a little earlier. We can find the rate of this effect by drawing a tangent to the curve of the Equation of Time at today's date. We may get result that the rate is -30 minutes in ~5 months or about -12 seconds per day.

So the the upper culmination of the Sun today will be at 11:29:43 – 00:00:12 = 11:29:31.

The difference in height of the Sun at the culminations yesterday and today is equal to the value of the change of the declination of the Sun during the day, $\Delta h = \Delta\varepsilon$. Easy approximation of the declination of the Sun for the current season is

$$\delta = -\varepsilon \cdot \sin B,$$

where $\varepsilon = 23°26'$, B is an analog of angle, $2\pi \sim 360° \sim 365.25^d$, starting at the moment of the autumn equinox (this year it was at about noon of September 23). Assuming argument was days, and N was the number of days since September 23,

$$\delta = -\varepsilon \cdot \sin(2\pi N/365.25),$$

and the daily variation is defined by the derivative of this formula:

$$d\delta / dN = -(2\pi/365.25)\, \varepsilon \cdot \cos(2\pi N/365.25),$$

N = 24 for today, $d\delta / dN = -(2\pi/365.25) \cdot \varepsilon \cdot \cos(2\pi N/365.25) = -0.37° \approx -22'$.

So the Sun today culminates 22' lower than yesterday.

αβ-2. Eclipse on the Poles. It seems that since the points of observations are opposite, the situation should be absolutely symmetric: the White Bear should also observe half of the eclipsed solar disk on the horizon at this time. But this quite incorrect.

It is necessary to take into account three important circumstances.

First, for any observer, the physical horizon is slightly lowered. For example, if a person stands on a flat surface, the depression of physical horizon she sees is about 2.5'. The depression of the physical horizon is easy to calculate: if R is the radius of the Earth, and h is the height of eye level above the surface of the Earth, the horizon dip is equal to

$$\arccos(R/(R+h)) \approx ((R+H)^2 - R^2)^{1/2}/R \approx (2h/R)^{1/2} \text{ (in radians)}.$$

Penguins may be different. For an estimation let us take the sizes of an Imperial Penguin, whose eyes' height above the surface of the Earth is about 1 m. The horizon depression is then about 2'. The White Bear, as it is mentioned, is sitting, so height of his eyes above the surface of the Earth will be about 1 m too, and so the horizon depression is about 2' too (If it stands up, this height would be more than 2 m and about 3' accordingly).

This means that if, as seen by the Penguin, the centre of the solar disk is on the horizon, this centre already is 2' below the mathematical horizon. So at the opposite point of the Earth – at the North Pole – it is 2' above the mathematical horizon. Accordingly, the sitting Bear-observer sees it already 4' above physical horizon, i.e. a quarter of the radius of the Sun.

But horizon depression is actually the smallest effect in this problem.

Secondly, there is refraction of light beams in the atmosphere. The value of a refraction depends slightly on weather conditions, but on average contributes with about 35' at the horizon. So at the moment when the centre of the solar disk is visible on the horizon at the South Pole, at the opposite point of the Earth – at the North Pole – the Sun may be seen quite above the horizon. In fact the refraction at the South Pole yields 35', and at North Pole about 25' (not 35', since the Sun not at the horizon but higher).

But this effect is also not the main one in the given problem.

The parallax of the Moon should be taken into account. It is no accident the annular eclipse may be observed only in a narrow band on the Earth surface. The distance to the Moon is 384 thous. km. The polar radius of the Earth is 12730 km. So the parallactic displacement of the Moon is 12730 / 38400 ≈ 0,0332 rad ≈ 1,9°.

That is, the Bear do not observe any solar eclipse this time, neither annular nor even partial.

Deduction: the White Bear sees the centre of the Sun above the horizon, at approximately 4'+35'+25' = 64' or about one degree. And the lunar disc is at approximately 1°54' below the solar one, i.e. totally under the horizon. (And it should be noted, 1°54' below the true position of the solar disk, while taking into account the refraction the true position of the Moon is 2°19' below the apparent position of the Sun).

αβ-3. Close conjunction. The corresponding scheme is drawn right on the right (**3.1.**). The values of angles are:

$$\alpha = \arccos(A_V/(1+e)A_M) = \arccos(0.723/1.093 \cdot 1.524) = 64.3°.$$

$$\beta = \arccos(A_V/A_E) = \arccos(0.723/1) = 43.7°.$$

$$\gamma = 180° - \alpha - \beta = \text{arcos}(A_V/A_E) = \arccos(0.723/1) = 72°.$$

Mars is in aphelion, which means it is opposite to the position of perihelion. We should remember that Mars in perihelion may be in great opposition for observers on the Earth in the late August (actually, August 25). So the current direction 1 (Mars→Sun) corresponds to the direction Sun→Earth for August 25. Direction 2 corresponds to the direction Sun→Earth 72° earlier, i.e. 71 day earlier, for June 15. By the sky map we may find that this direction is to a point in Ophiuchus constellation just near to the border with Sagittarius (**3.3.**).

The above conclusion means also that this situation was released in the middle of June. At this time the Sun is shining for the Bear at the North pole, and the Moon (that is in the opposite point of the celestial sphere at the moment of eclipse) may be observed by Penguin at the South pole. (**3.2. B– P+**).

3.4. To find the period from the situation from the previous problem to the current situation, we should take into account that the first situation was near equinox, and the second – in the middle of June. And both of them are possible only with the Moon is near its orbital nodes. The orbital nodes of the Moon move in opposite direction with period 18.6 years. That is the minimum possible time between the two situations may be in the case the line of nodes moved from end September – end March line to middle June – middle December line, that is about 28% of the period or 18.6×0.28 ≈ 5.2 years. So the minimum time is about 5 years and almost 2-3 months later. The number of synodic periods of the Moon in 5 years and exactly 3 months is (365.25×5.25)/29.53 = 64.94. For our situations this value should be an integer and a half. 64.5×29.53 = 1904.7.

Let us check. The situation very near the first condition was just on March 20, 2015 (total solar eclipse very close to North pole and just at the day of equinox). 1904.7 days later is June 5-6, 2020. Yes! The lunar eclipse (although not full but partial) will occur in the evening of June 5. And the Moon will be in the constellation of Sagittarius.

α-4. **Alpha Centauri.** The magnitude of the Sun is $M_{Sun} = -26.8^m$, the magnitude of Alpha Centauri A – $m_{\alpha CenA} = -0{,}01^m$, Alpha Centauri B – $m_{\alpha CenB} = 1{,}33^m$. Comparing these values one can calculate, from which star more light comes to us, and how many times more. But we can not know, which of the objects emits more, because they are at different distances.

Let us calculate the apparent magnitude of Alpha Centauri A, if it was located at the place of the Sun,

$$M_{\alpha CenA} = m_{\alpha CenA} + 5\cdot\log((pc/au)''/p_{\alpha Cen}) = -0{,}01^m + 5^m\cdot\log(206265''/0.747'') \approx -0{,}01^m - 27.21^m = -27.22^m.$$

This value is smaller than that of the Sun. Then Alpha Centauri A radiates more energy. Adding radiation Alpha Centauri B (not even calculating the value), gets more. Thus, Alpha Centauri A+B emits more energy than the Sun.

β-4. **The Constellation White Leopard.** There are 24 letters in Greek alphabet. So we have to take into account 24 magnitudes from $+0.10^m$ till $+2.40^m$.

[$0.10^m + 0.20^m + 0.30^m + \ldots + 2.40^m = 12 \times 2.50^m = 30^m$. Sorry, a joke.]

Really, we have to add to the flux from the star with the magnitude 0.10^m fluxes from other 23 stars, the values of which decrease exponentially with a pitch

$$10^{0.4\cdot 0.10} \approx 1.09648$$

The total flux $\Phi(1\text{-}\infty)$ of infinite series of such values can be calculated as the sum of an infinite geometric progression, or from the equations

$$\Phi(1\text{-}\infty) = \Phi(1) + \Phi(\infty)/1.09648$$

$$\text{and} \quad \Phi(1\text{-}\infty) = \Phi(\infty)/1.04713 = \Phi(1).$$

$$\Phi(1\text{-}\infty) = \Phi(1)/(1 - 1/1.04713) = \Phi(1) \times 11.37.$$

The total magnitude m (1-∞) an infinite number of stars would be equal to:

$$m(1\text{-}\infty) = 0.10^m - 2.5^m \log 11.37 \approx -2{,}54^m$$

However, there are not an infinite number of stars but only 24 in the constellation of the White Leopard. To obtain the result we have to subtract from the flux from the infinite series starting with 0.10^m, the flux from the analogous infinite series but starting with the 25th star, i.e. 2.50^m. As each star is the second series is 2.40^m weaker than in the first one, the result for the flux from the second series will also be on 2.40^m weaker.

$$m(25\text{-}\infty) = 2.50^m - 2.5^m \log 11.37 \approx -0.14^m.$$

Then we need to find the difference between -2.54^m and -0.14^m.

[$-2.54^m - -0.14^m = -2.40^m$. Sorry, a joke once again.]

The flux $\Phi(25\text{-}\infty)$ is $10^{0.4\cdot 2.4} \approx 9.12$ times weaker than the flux $\Phi(1\text{-}\infty)$.

$$\Phi(25\text{-}\infty) = \Phi(1\text{-}\infty) / 10^{0.96} = \Phi(1) \times 11.37 / 9.12 \approx \Phi(1) \times 1.246.$$

Thus, the flux from 24 stars of the constellation will be equal to

$$\Phi(1\text{-}24) = \Phi(1\text{-}\infty) - \Phi(25\text{-}\infty) = \Phi(1) \times (11.37 - 1.246) = \Phi(1) \times 10.12.$$

The total magnitude m(1-24) of all of these stars is:

$$m(1\text{-}24) = 0.10^m - 2.5^m \log 10.12 \approx -2.41^m.$$

*Attention! Joking aside, here is our second joke (**$-2{,}54^m - -0{,}14^m = -2{,}40^m$**) gives almost the right numerical answer. It is pure coincidence, clearly shows that not any solution resulting in the same numerical value is right. Pay attention when checking!*

Note. If the inability to follow properly such (or similar) logic, there can be found such options:

– To calculate head-on the flux of each star, sum the fluxes to get the total flux, and find from it the total apparent magnitude. With the proper calculation of the answer is exactly the same (one may easily check it in Excel). But it is too long to calculate it without a computer or a programmable calculator.

– To estimate very rough, using a linear approximation of magnitude depending on the flux. The "Average" magnitude is $(0.10^m + 2.40^m) / 2 = 1.25^m$. The total magnitude of these 24 stars is

$$1.25^m - 2.5^m \log 24 \approx 1.25^m - 3.45^m = -2.20^m.$$

(This is a very rough and not suitable as a workable solution, but it's a little better than nothing).

α-5. **Motion of a satellite.** To solve the problem let's work in the frame of reference rotating around the Earth along with the satellite. In this frame the satellite is affected by two forces: the centrifugal force $F_c = mV_1^2/R$ and the gravitational force $F_g = GMm/R^2 = mV_0^2/R$, where V_0 and V_1 are the circular and increased by 0.6% velocities of the satellite at the height H = 428 km, and $R = R_E + H$ – the radius of this orbit. Just we see that in this frame of reference the first minutes after the perigee passage the satellite simply moved from the Earth with a constant acceleration:

$$a = (F_c - F_g)/m = (V_1^2 - V_0^2)/R = (1{,}006^2 - 1^2)\, g_1,$$ where g_1 is the acceleration of gravity at the height H.

$$a = 0{,}012 \cdot g_1 = 0{,}012\, g_3\, [R_3 / (R_3+H)]^2 \approx 0{,}103 \text{ m/s}^2.$$

Then from the formula $\Delta H = at^2/2$ the time to reach the height H_1 may be found, $t = (2\Delta H/a)^{1/2}$, which is about 1160 seconds or 19.4 minutes. It is much smaller than the period (about 95 minutes), but already comparable with the quarter of the period. So this result is estimation.

Note: As an exact solution this method is applicable only to a small part of the orbit of the satellite after the perigee passage, in the case of t << T (the period of the satellite).

β-5. **Spiral galaxy.** At first, we shall notice, that the values in the problem are given with an accuracy of one digit only, moreover, we will need the difference between the wavelengths, which can be obtained with only one digit too, therefore a final answer with greater accuracy is inappropriate. The redshift of the galaxy is equal to $z = \Delta\lambda/\lambda \approx 0.075$, which means, that it is receding us with a speed $v = z \cdot c = 22\,500$ km/s. According to Hubble's law it is located on distance L = v/H. Taking Hubble's constant equal to H = 68 km/c /Mpc,

$$L = zc/H \approx 320 \text{ Mpc}.$$

The angular size of the galaxy is 40", that is the real size is

$$D = (40/205265) \cdot 320 \text{ Mpc} \approx 62 \text{ kpc} \approx 1.9 \cdot 10^{21} \text{ m}.$$

$$R = D/2 = 31 \text{ kpc} \approx 1.0 \cdot 10^{21} \text{ m}.$$

As the spiral galaxy is visible not as a circle but as an ellipse, its plane is inclined to line sight on angle α,

$$\cos \alpha = 30/40, \quad \alpha \approx 41°.$$

The broadening of the spectral lines is due to motions of stars. The range from 6854 Å to 6857 Å means a broadening of 1.5 Å in each direction, that is, the radial velocity of the stars that are on the edge of the galaxy, is $V_R = (1.5 \text{ Å} / 7055 \text{ Å}) \cdot c \approx 64$ km/s. Their total velocity in space is

$$V = V_R / \sin \alpha \approx 96 \text{ km/s}$$

To estimate the mass of the galaxy we can assume that the most remote stars in the galaxy are moving in circular orbits with this velocity V according to Kepler's laws, and the mass of the galaxy M is concentrated in its center. Thus

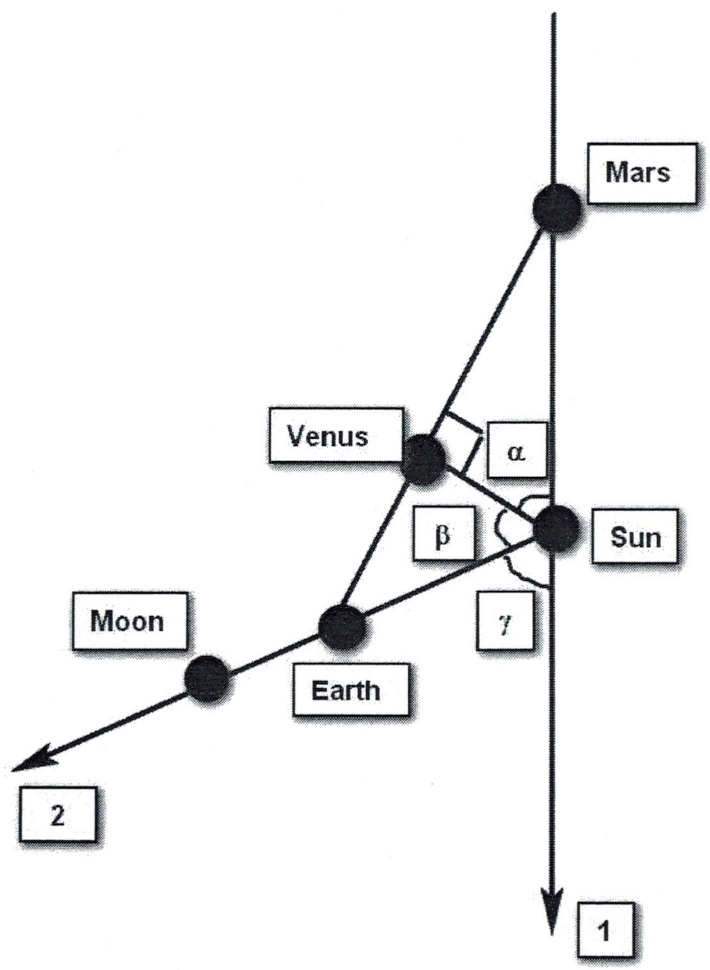
$$V^2/R = GM/R^2,$$

$$M = V^2R/G = 1.34 \cdot 10^{41} \text{ kg}.$$

Thus, if the galaxy consisted of stars of spectral type A7-A8, let us take Altair as the typical star of the galaxy. The mass of Altair is 1.7 of solar masses, i.e. $3.4 \cdot 10^{30}$ kg.

$$N = M/M_A \approx 4 \cdot 10^{10},$$

that is, the order of 40 billion stars.


Mars
Venus
α
β
Sun
γ
Moon
Earth
2
1