Theoretical round. Sketches for solutions

Note for jury and team leaders. The proposed sketches are not full; the team leaders have to give more detailed explanations to students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

αβ-1. Transit of Venus. Any transit of Venus may occur only in configuration of inferior conjunction of Venus. To calculate next inferior conjunctions we need first to find the synodic period T_S of Venus:

$$1/T_S = 1/T_V - 1/T_E,$$

where T_V and T_E are the sidereal periods of Venus and Earth respectively,

$$T_S = T_E \cdot T_V / (T_E - T_V) = 583.92^d.$$

January 1, 2117 will be in 105·365 + (26-1) – (5+31+30+31+28+31) = 38194 days after June 6, 2012. December 31, 2017, respectively, will be in 38194+364 = 38558 days after that date.

$$38194 / 583.92 = 65.41.$$
$$38558 / 583.92 = 66.03.$$

It means that the close approach of Venus in 2117 will be after 66 synodic periods of Venus.

$$66 \times 583.92 = 38538.7.$$

38538 days are 20 days earlier than the date 38558 that we calculated for December 31, 2117. Thus, 38538 days after June 6, 2012 corresponds to December 11, 2117.
Answer: The next transit of Venus will take place on December 11, 2117.

αβ-2. Transit of Pseudovenus. Visible motion of Venus depends on the synodic motion of the planet. During the Venus transit we see just synodic velocity of Venus. The synodic angular speed of Venus is:

$$\omega = \omega_V - \omega_E,$$

where ω_V and ω_E are angular sidereal velocities of Venus and Earth respectively. If R_V and R_E are the radii of the orbits, the velocity of Venus in this system is,

$$V = \omega \cdot R_V = (\omega_V - \omega_E) \cdot R_V,$$

and its visible angular speed on the Earth's sky,

$$u = V / (R_E - R_V) = (\omega_V - \omega_E) \cdot R_V / (R_E - R_V).$$

But the synodic motion of the Pseudovenus has the direction opposite to that of planet's rotation around the Sun. So the relation between the synodic **u** and sidereal ω_P angular velocities of Pseudovenus will be

$$u = \omega_P + \omega_E$$

(opposite to formula $u = \omega_P - \omega_E$ for the same direction). So

$$\omega_P = u - \omega_E,$$
$$\omega_P = (\omega_V - \omega_E) \cdot R_V / (R_E - R_V) - \omega_E,$$
$$\omega_P = \omega_V \cdot R_V / (R_E - R_V) - \omega_E \cdot (R_V / (R_E - R_V) + 1) = \omega_V \cdot R_V / (R_E - R_V) - \omega_E \cdot R_E / (R_E - R_V) =$$
$$= (\omega_V \cdot R_V - \omega_E \cdot R_E) / (R_E - R_V) = 2\pi \cdot (R_V/T_V - R_E/T_E) / (R_E - R_V).$$

For a body rotating around the Earth (mass M) in a circular orbit we may write,

$$\omega^2 R = GM/R^2,$$
$$R^3 = GM_E/\omega^2,$$
$$R_P = [GM_E(R_E - R_V)^2 / 4\pi^2 \cdot (R_V/T_V - R_E/T_E)^2]^{1/3}.$$

Calculations give us,

$$R_P = 2.92 \cdot 10^9 \text{ m} = 2.92 \text{ mln.km (approx. 7.60 radii of lunar orbit).}$$

There is also an other way to get the result after we have ω_P. It is to compare the motion of Pseudovenus with the motion of Moon. Since according to III Kepler law, T is proportional to $R^{3/2}$, ω is proportional to $R^{-3/2}$,

$$(\omega_P/\omega_M) = (R_P/R_M)^{-3/2},$$
$$R_P = R_M \cdot (\omega_M/\omega_P)^{2/3},$$
$$R_P = R_M \cdot (2\pi/T_M\omega_P)^{2/3}.$$

And calculations give us,

$$R_P = 7.62 \ R_M = 2.93 \text{ mln.km.}$$

Note for jury. Additional point may be given for the students who note that this distance is out of the Hill sphere for the Earth and so this situation is impossible.

We may find the size (diameter) of Pseudovenus using this distance R_P and the angular size of the object that is equal to:

$$\alpha = r_V/(R_E - R_V),$$

where r_V is the diameter of the real Venus.

$$r_P = \alpha \cdot R_P = r_V \cdot R_P/(R_E - R_V).$$

Calculations give us,

$$r_P = 8.55 \cdot 10^5 \text{ m} \approx 850 \text{ km (approx. 0.25 diameters of the Moon).}$$

αβ-3. Old persons' star. Canopus is a star of the southern sky; if it is visible in Korea, it should be close to the horizon. Absorption and scattering of light play an important role under such conditions of observations. Therefore, the most favorable conditions for the observation are at the southern point of the island Jeju, and Canopus is at upper culmination. The latitude of this point is the smallest of the given range, i.e. 33°12' N. At this latitude Canopus culminates at the altitude

$$\mathbf{h} = 90° - \varphi + \delta = 90° - 33°12' + -52°42' = 4°06'.$$

Atmospheric absorption and scattering are significant for observations at such a low altitude. The depth of atmosphere which a beam of light from the star passes is 1/sin**h** times larger than the depth that the light from a star located in zenith passes. $1 / \sin 4°06' \approx 14$. (Otherwise, one may use the formal formula using the zenith angle $1 / \cos z = 1 / \cos 85°54' \approx 14$.) It is known that under the most favorable conditions, the loss of light when it passes one atmosphere (due to absorption and scattering) is about 20% (or 19% as it is written in the supplement table), or (in magnitudes) $0^m.23$. Since the magnitudes of stars in the table ($\mathbf{m_0} = -0^m.72$ for Canopus) are given with the zenith absorption and scattering ($+0^m.23$ to the magnitude visible from space), the additional absorption of light at a height of 4°06 'will be $(1 / \tan 4°06' - 1) \approx 13$ times more than in zenith. Thus,

$$\Delta\mathbf{m} = 0^m{,}23\ (1 / \sin 4°06' - 1) \approx 3^m.0.$$
$$\mathbf{m_1} = \mathbf{m_0} + \Delta\mathbf{m} = -0^m{,}72 + 3^m{,}0 \approx 2^m.3.$$

Note for jury. This problem is to estimate. So the exact values are not too important, especially knowledge the exact value of 20% (15% – 30% may be reasonable), or (in magnitudes) $0^m.23$ ($0^m.15 - 0^m.4$ may be reasonable). And the reasonable answers are from $1^m.5$ to 4^m.

α-4. Stars on Mars. As we may see from the "Data of some stars", Canopus and Sirius have almost the same RA (Earth based coordinates) while the difference in DEC is significant. It means that the distance between these stars in the Earth sky is approximately equal to the difference in DEC,

$$\beta = \delta_1 - \delta_2 = (-16°42'58'') - (-52°41'45'') \approx 35°59' \approx 36°.$$

After moving the observer to Mars, this angle will not change, although the coordinates in the Martian sky will be different. It means, the distance measured, by along Martian surface, between the points where Canopus and Sirius are in zenith is:

$$\beta(\text{rad})\times R = \beta(°)\times\pi R/180 \approx 2130 \text{ km}.$$

where R = 3397 km is the radius of Mars.

If the Bear sees Canopus in zenith, the zenith angle of Sirius will be $\beta \approx 36°$, and its height above horizon

$$\mathbf{h} = 90° - \beta \approx 90° - 36° = 54'.$$

Answers: ~2300 km, 54°.

β-4. Altair. All necessary data may be taken from tables and the Hertzsprung-Russell diagram.
Distance to Altair is

$$D_A = 1 \text{ pc} / \boldsymbol{p} = 206265 \text{ a.u.} / 0.195 \approx 1\,060\,000 \text{ a.u.}$$

Our Sun being moved to this distance will have the magnitude

$$\mathbf{m_1} = -26^m.74 + 5^m \lg 1\,060\,000 \approx -26^m.74 + 30^m.12 \approx 3^m.38.$$

So the difference in absolute stellar magnitudes of Sun and Altair is:

$$\Delta\mathbf{m} = 3^m.38 - 0^m.77 = 2^m.61.$$

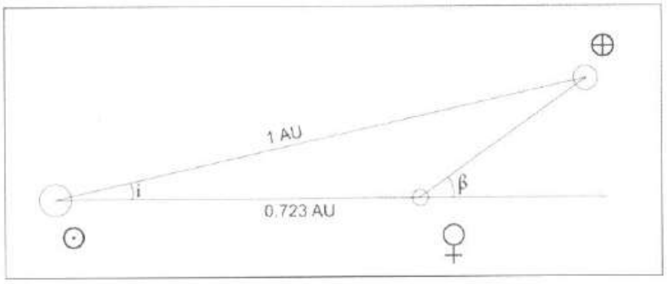
Thus, the ratio of luminosities of Altair L_A and the Sun L_0 is

$$L_A / L_0 = 100^{2,61/5} \approx 11.1.$$

The luminosity of a star L is proportional to its surface area and the fourth power of surface temperature, i.e. $L \sim R^2T^4$. Density of a star is equal to its mass divided by its volume, i.e. it is proportional to M/R^3. Thus, $\rho \sim MT^6/L^{3/2}$. Temperatures of Altair and Sun may be found using the Hertzsprung-Russell diagram from their spectral types, A7 and G2, 8100 K and 5800 K respectively. So by comparing the densities of Altair and Sun (see table, $\rho_0 \approx 1410$ kg/m^3), we may find that the density of Altair is

$$\rho_P = \rho_0\cdot(M_A/M_0)\cdot(T_A/T_0)^6\cdot(L_0/L_A)^{3/2} \approx 480 \text{ kg/m}^3.$$

α-5. Venus and Earth. The maximum space distance from the Earth to the Venus ecliptic is (see figure)



$$H = R_0 \times \sin \boldsymbol{i},$$

where R_0 is radius of the Earth orbit (1 AU), and $\boldsymbol{i}$ is the orbital inclination of the terrestrial orbit to the plane of Venus ecliptic, which is evidently equal to the orbital inclination of the Venus orbit to the plane of the Earth's ecliptic, i.e. 3.4°.

$$H = 1 \text{ AU} \times \sin 3.4° = 0.0593 \text{ AU},$$

In the Venetian sky, the Earth in such position can be visible at the maximum distance at the configuration of opposition, when the distance between Venus and Earth is minimal and equals to

$$L = R_0 - R_V = 1 \text{ AU} - 0.723 \text{ AU} = 0.277 \text{ AU}$$

(all angles are small and we do not take into account the inclination in these calculations). The maximal distance in the sky (angle $\boldsymbol{\beta}$) may be found from the equation

$$H = L \times \tan \boldsymbol{\beta},$$
$$\text{tg}\, \boldsymbol{\beta} = H / L = 0.0593 \text{ AU} / 0.277 \text{ AU} = 0{,}214,$$
$$\boldsymbol{\beta} = 12{,}1°.$$

β-5. Venus and Earth. First part of the solution, see solution of problem **α-5**.

To estimate stellar magnitude of the Earth visible from vicinities of Venus in opposition we may compare it with Mars visible from the Earth in opposition (below α is albedo, D are diameters of the bodies and R are distances, indices E, V, M and S correspond to Earth, Venus, Mars and Sun).

The flux to Venus from Earth $\qquad F_V \sim \alpha_E\cdot D_E^2\cdot(1/R_{V\text{-}E})^2\cdot(1/R_{S\text{-}E})^2.$

The flux to Earth from Mars $F_M \sim \alpha_M \cdot D_M^2 \cdot (1/R_{E\text{-}M})^2 \cdot (1/R_{S\text{-}M})^2$.

The ratio of fluxes $F_E/F_M = (\alpha_E/\alpha_M)\cdot(D_E^2/D_M^2)\cdot(R_{E\text{-}M}\cdot R_{S\text{-}M})^2/(R_{V\text{-}E}\cdot R_{S\text{-}E})^2$.

Taking the necessary values from the table of Solar system, we may calculate:

$$F_E/F_M = (0.36/0.15)\cdot(12756/6794)^2\cdot(0.524\cdot 1.524)^2/(0.277\cdot 1)^2.$$
$$F_E/F_M \approx 70.$$

So the Earth visible from vicinities of Venus at opposition is brighter than Mars visible from Earth at opposition, and the difference in stellar magnitudes is equal

$$\Delta m = -2^m.5\cdot \lg(F_E/F_M) \approx -4^m.6.$$

Stellar magnitude of the Earth

$$m = -2^m.0 + \Delta m \approx -6^m.6.$$

Of course it is not the only possible correct way for solution.

α-6. Parallaxes. 6 light years is equal to (6/3.26) pc ≈ 1.84 pc.
A parallax of 0.001" corresponds to a distance of 1000 pc.
So the interferometer cannot measure the parallaxes of the stars that are more distant than 1000 pc with any reasonable accuracy.
If we had a uniform distribution of stars in space, the volume of space with these stars could be considered a sphere with radius of 1000 pc. For the number of stars one would write:

$$N = 4\pi/3\ (1000\text{ pc} / 1.84\text{ pc})^3 \approx 6.7\cdot 10^8.$$

or, with reasonable accuracy to one significant digit, $7\cdot 10^8$ stars, i.e. seven hundred million stars.
However, the stars of our Galaxy are distributed not evenly in all directions. In our part of the Galaxy the thickness of the Galaxy is less than 1000 pc, but is only about 400 pc. Therefore, the volume of space with these stars could be considered a cylinder with a radius of 1000 pc and a height of 400 pc. The number of stars in this volume:

$$N = \pi\ (1000\text{ pc} / 1.84\text{ pc})^2(400\text{ pc} / 1{,}84\text{ pc}) \approx 2\cdot 10^8$$

or two hundred million stars.

Note for jury. This problem is to estimate. So the exact values are not too important, especially knowledge of the exact value of the thickness of Galaxy. In this point the most important is simple understanding that this effect should be taken into account, and reasonable order of value of the thickness of Galaxy.

β-6. Remote galaxy. The fact that the galaxy, consisting of yellow stars like the Sun, looks like the orange star **ε** Eridani indicates redshift. $z = (\lambda_1 - \lambda_0)/\lambda_0$. According to Wien's displacement law $\lambda T = \text{const}$. $\lambda_1 T_1 = \lambda_0 T_0$. Therefore, $z = (T_0 - T_1)/T_1$.

The approximate temperatures of the stars may be found from the Hertzsprung-Russell diagram,
$T_0 \approx 5800$ K, $T_1 \approx 4900$ K.
Hence $z \approx (5800 - 4900)/4900 \approx 0.18$.
The galaxy is receding from us with a speed $V = zc$. $V = 0.18\cdot 300000$ km/s = 54000 km/s.
According to Hubble's law, $V = R\cdot H$, $R = V/H$ = 54000 km/s / 71 km/s/Mpc ≈ 760 Mpc.

Thus, the galaxy is located at the distance 760 Mpc, and the fact that it looks like ε Eridani ($3^m.74^m$), weakened in brightness by factor 1000 ($3^m.74^m + 7^m.5^m = 10^m.24^m$) allow us to conclude that (excluding the effect of the redshift) the absolute magnitude of the galaxy is:

$M = m - 5^m\cdot\log(R/10\text{ pc}) = 3^m.74 + 7^m.5 - 5^m\cdot\log(76\,000\,000) \approx -28^m.2$.

Redshift leads to the fact that every photon coming to us loses some of the energy, $E = h\nu = hc/\lambda$, $\Delta E = h\Delta\nu = hc(1/\lambda_0 - 1/\lambda_1)$, $\Delta E/E = \Delta T/T_0 \approx 0.16$, $E_1/E_0 = T_1/T_0 \approx 0.84$. This change the magnitude by $\Delta m = -2^m.5\cdot\lg(0.84) \approx 0^m.2$.

Thus, taking into account the effect of redshift, absolute magnitude of the galaxy is

$M = -28^m.2^m + -0^m.2^m \approx -28^m.4^m$.

The absolute magnitude of the Sun is $4^m.8$, the difference is $\Delta M = 33^m.2$.

Using this value we can conclude that the total number of stars is $10^{\Delta M/2.5} \approx 10^{13.28} \approx 1.9\cdot 10^{13}$ or about 20 trillion stars.