АСТРОНОМИЧЕСКОЕ ОБЩЕСТВО

EURO–ASIAN ASTRONOMICAL SOCIETY

Round Theo

Group α β

XV Международная астрономическая олимпиада

XV International Astronomy Olympiad

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язык language **English**

Theoretical round. Sketches for solutions

Note for jury and team leaders. The proposed sketches are not full; the team leaders have to give more detailed explanations for students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

αβ-1. **Circumpolar stars.** Obviously, the problem is of estimation. So we will not take into account the effect of diminishing the number of visible stars due to weak transparency of the atmosphere near the horizon. Also, at first we will consider that stars are distributed more or less regularly in the sky.

1.1. The number of circumpolar stars at the North pole equals to a half of 6000, i.e. 3000. It is those stars whose declination is above 0°. Due to refraction the number of circumpolar stars is increased by those ones whose declination is below 0° but above -35'. Let us estimate the ratio between the number of stars which are located in the range of declinations from -35' to 0' to the number of stars of all the sphere. It is the same as the ratio between the solid angle of the belt between the declinations -35' and 0' and the solid angle of whole sphere (4π). [It is good to draw a figure.]

It is more visual not to use solid angles but areas on an imagined sphere of radius R_0, and calculate the above ratio as the ratio between the area of the belt near the celestial equator with 35' width and the area of the whole sphere ($4\pi R_0^2$). The area of the belt is its length along the celestial equator ($2\pi R_0$,) multiplied by its width $R_0 \cdot 35 \cdot \pi/(180 \cdot 60)$ (where $\pi/(180 \cdot 60)$ is the value of an arc minute expressed in radians). The ratio between these areas is

$$2\pi R_0 \cdot R_0 \cdot 35 \cdot \pi/(180 \cdot 60) \,/\, 4\pi R_0^2 = 35\pi/21600.$$

The number of stars in this belt may be estimated as

$$6000 \cdot 35\pi/21600 \approx 30.5,$$

but the accuracy even to one star here is obviously excessive, therefore it is necessary to say that about 30 stars become circumpolar due to refraction.

One may use a similar procedure to find the answer for the question **1.2.** of the problem. Due to refraction the stars placed closer 35' to the poles (both Northern, and Southern), that is, with declinations above +89°25' or below -89°25', become circumpolar. The ratio between the area of these slices of imagined heavenly sphere (area of each slice is $\pi \cdot (35')^2 = \pi \cdot (35\pi/(180 \cdot 60))^2$) and the area of the whole sphere 4π is $5.2 \cdot 10^{-5}$. By multiplying this value by 6000 we may get approximately 0.31 star. *[And some participants just have written: there will be about 0.3 new circumpolar stars.]* But what does this value mean? It means that the model of uniform distribution of stars on heavenly sphere does not work any more. Any stars ("one star", "two stars" etc.) may be present in these areas of the sky, or may not be present, but it can not be "half present". And the right answer may be given at once (without any calculations!): **no star**, as we know that there are no stars visible with the naked eye star in radius 35' near to poles. The nearest one is Polaris; it is situated approximately 45' from the North Pole.

αβ-2. **Observation of a star.** Light from a star at a height of 45° above the horizon passes layers of atmosphere at an angle of 45°, that is, the thickness of each layer will be 1.414 times larger. That is, light will pass effectively 1.414 thickness of atmosphere. Thus, $1.414 - 1 = 0.414$ part of all atmosphere increases the stellar magnitude by $\Delta m_{21} = m_2 - m_1 = 2.85^m - 2.74^m = 0.11^m$. It means, the whole of atmosphere increases the stellar magnitude by $\Delta m = 0.11^m/0.414 \approx 0.27^m$. The stellar magnitude of the star observed from above the atmosphere will be less by this Δm value from the value measured while observed at zenith from the Earth,

$$m_0 = m_1 - \Delta m = 2.74^m - 0.27^m = 2.47^m.$$

α-3. Parallax. From the table "Data of some stars" we can find that the distance to Sirius is

$$L = 1/p \text{ pc} = 1/0.379 \text{ pc} = 2.64 \text{ pc} = 206\,265 \times 2.64 \text{ a.u.} \approx 5.44 \cdot 10^5 \text{ a.u.}$$

This value equals to 1.406 mepc, that is

$$1 \text{ mepc} = 206\,265 \text{ a.u.} / 0.379 / 1.406 = 5.44 \cdot 10^5 \text{ a.u.} / 1.406 \approx 3.87 \cdot 10^5 \text{ a.u.}$$

But $3.87 \cdot 10^5$ a.u. is equal to 10^6 semiaxis of orbit of Mercury. So it is reasonable to propose that astronomers of Mercury measure annual parallax in microradians and use this mercurial angular unit for definition of "their parsec". So the mercurial parsec is really par-μ-rad.

The mercurial horizontal (diurnal) parallax of the Sun is the ratio of equatorial radius of Mercury and the average distance to Sun

$$p_M(\text{sun}) = r_M/a_M = d_M/2a_M = 4879 \text{ km} / 2 \cdot 57\,900\,000 \text{ km} \approx 40.9 \cdot 10^{-6} \text{ rad.}$$

That is $p_M(\text{sun}) = 40.9$ mercurial angular units.

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This value equals to 19.6 vpc, that is

$$1 \text{ vpc} = 206\,265 \text{ a.u.} / 0.379 / 1.406 = 5.44 \cdot 10^5 \text{ a.u.} / 19.6 \approx 2.78 \cdot 10^4 \text{ a.u.}$$

This distance is equal to $2.78 \cdot 10^4 / 0.723 = 3.84 \cdot 10^4$ semiaxis of orbit of Venus.

By logic we should propose that Venusials use tetradecimal (base-14) numeral system, so let us find $\log_{14}(3.84 \cdot 10^4)$.

$$\log_{14} 38\,400 = \ln 38\,400 / \ln 14 = 3.996 \approx 4,$$

So $$3.84 \cdot 10^4 \approx 14^4.$$

(In fact exactly $38\,416 = 14^4$.) So it is reasonable to propose that astronomers of Venus use angle values in radians (rad), units similar to our centiradians (14^{-2} rad) and units 14^{-4} rad, and use the angular unit of 14^{-4}-radians (maybe called as microradians but "micro-" at Venus means 14^{-4}) for definition of "their parsec". So the venusial parsec is really par-14^{-4}-rad.

The venusial horizontal (diurnal) parallax of the Sun is the ratio of equatorial radius of Venus and the average distance to Sun

$$p_V(\text{sun}) = r_V/a_V = d_V/2a_V = 12\,104 \text{ km} / 2 \cdot 108\,200\,000 \text{ km} \approx 55.933 \cdot 10^{-6} \text{ rad.}$$

That is $p_V(\text{sun}) = 55.9 \cdot 10^{-6} \text{ rad} / 14^{-4} \text{ rad} = 2.15$ venusial angular units.

Note for jury. Some students may try to write the answer using tetradecimal (base-14) numeral system, which is not necessary, and answers in both decimal and tetradecimal system should be considered as correct. Answer in tetradecimal system may be found as follows:

In decimal system $2.1487 = 2 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 8 \cdot 10^{-3} + 7 \cdot 10^{-4}$.
Hence it equals tetradecimal value $K.LMN = K + L \cdot 14^{-1} + M \cdot 14^{-2} + N \cdot 14^{-3} + \ldots$

$$K = [2.1487] = 2, \quad L = [(2.1487\text{-}K)/14^{-1}] = 2, \quad M = [(2.1487\text{-}K\text{-}L \cdot 14^{-1})/14^{-2}] = 1,$$

$$N = [(2.1487\text{-}K\text{-}L \cdot 14^{-1}\text{-}M \cdot 14^{-2})/14^{-3}] = 2\ldots \qquad ([X] \text{ means integer part of } X.)$$

That is, the value written in decimal numeral system as 2.1487 will be 2.212 in tetradecimal system.

α-4. Climate. Solar (or stellar) energy, reaching to a planet in finally reemitted by the planet according to Stefan-Boltzmann law. We should notice before the main part of solution that the answer strongly depends on a relation of heat capacity of a planet and the period of its sidereal period. If the thermal capacity is large, and the period is small, then obviously, temperature on a planet varies a little even with large eccentricity of orbit. On the other hand, if the heat capacity is small, and the period is large, it is possible to consider that the planet is in thermal balance in each point of its orbit.

Let us consider the second case. L – luminosity of the Sun, r – the radius of the planet, T_W – winter temperature, T_S – summer temperature. For the planet in aphelion (distance to the Sun – R_A) the energy balance is,

$$(L / 4\pi R_A^2) \times \pi r^2 = \sigma T_W^4 \times 4\pi r^2,$$

And for the planet in perihelion (distance to the Sun – R_P),

$$(L / 4\pi R_P^2) \times \pi r^2 = \sigma T_S^4 \times 4\pi r^2.$$

From these two formulae,

$$R_A \times T_W^2 = R_P \times T_S^2,$$

$$R_A / R_P = (T_S / T_W)^2.$$

$$e = (R_A - R_P) / (R_A + R_P) = (T_S^2 - T_W^2) / (T_S^2 + T_W^2).$$

Numerical calculation (the temperatures expressed in Kelvin) gives us,

$$e = (294.4^2 - 272.8^2) / (294.4^2 + 272.8^2) = 0.076.$$

That is, such eccentricity of the orbit of the planet with small heat capacity will result for the temperature changes winter-summer similar to Crimean ones. However, for planets of type of the Earth temperature fluctuations will smooth out. That is, for the given conditions for the Earth like planet the eccentricity should be much larger. Thus, the answer

$$e = (T_S^2 - T_W^2) / (T_S^2 + T_W^2) = 0.076 \text{ is the minimum estimation.}$$

β-4. White dwarf. *The problem is devoted the 100^th anniversary of Subrahmanyan Chandrasekhar (Субраманьян Чандраксекар), which will be tomorrow, on October 19, 2010.*

For the planet (P) similar to the Earth on all parameters, it is necessary for it to receive from the white dwarf (WD) as much energy as the Earth (E) receives from the Sun (S):

$$E_P = E_E.$$

The energy received by a planet from a star is proportional to luminosity (power of radiation) of the star and inversely proportional to square of average distance to a star:

$$E \sim J / R^2.$$

For the Earth:

$$E_E \sim J_S / R_E^2.$$

For a planet rotating around the white dwarf:

$$E_P \sim J_{WD} / R_P^2.$$

Thus,

$$J_S / R_E^2 = J_{WD} / R_P^2,$$

whence

$$R_P = R_E \times (J_{WD}/J_S)^{1/2}.$$

The ratio of luminosities can be found from the difference of absolute stellar magnitudes,

$$J_{WD}/J_S = 10^{-0.4(Mwd-Ms)}$$

Thus,

$$R_P = R_E \times (10^{-0.4(Mwd-Ms)})^{1/2} = R_E \times 10^{-0.2(Mwd-Ms)}.$$

The absolute stellar magnitude of the Sun M_S should be known by a participant of the Olympiad, or it can be calculated that the value of $m_S = -26.8^m$ of the visible stellar magnitude, from the table and formula for distance of 10 pc written in a.u. (10 pc = 2 062 648 a.u.):

$$M_S = m_S + 5^m \cdot \lg 2062648 = -26.8^m + 31.57^m = 4.77^m.$$

According to general III Kepler law,

$$T^2M/R^3 = \text{const} \qquad \text{therefore} \qquad T_P = T_E(R_P/R_E)^{3/2}(M_{WD}/M_S)^{-1/2}.$$

We know the values T_E and R_P/R_E, then what may be the value of M_S/M_{WD}? Let us remember the Chandrasekhar (*) limit, that is equal to 1.4 M_S – the upper limit for mass of a star that may exist as a white dwarf.

(*) Subrahmanyan Chandrasekhar (Субраманьян Чандраксекар) – American scientist of the Indian origin whose 100 anniversary will be tomorrow, on October 19, 2010.

Thus, the minimal sidereal period is equal

$$T_P = T_E(10^{-0.2(Mwd-Ms)})^{3/2}(1.4)^{-1/2} = T_E \times 10^{-0.3(Mwd-Ms)} \times (1.4)^{-1/2} \approx 0.53 \text{ days}.$$

α-5. Cosmonaut. For estimation we may consider the space station and cosmonaut as pointlike particles. The time $\tau_{1/2}$ after which these particles collide due to gravitational attraction is approximately equal to half of period of revolution of mass m around mass M on the orbit with the major semiaxis L/2. And the time τ that we should find approximately equals $\tau_{1/2}$ (neglecting the last part of orbit comparable with the sizes of station and cosmonaut). According to general III Kepler law,

$$4\pi^2/GM = T^2/(L/2)^3, \quad \text{so} \quad T = 2\pi(L/2)^{3/2}(GM)^{-1/2}, \quad \text{and} \quad \tau_{1/2} = T/2 = \pi(L^3/8GM)^{1/2}$$

$$\tau \approx \tau_{1/2} = 4.35 \cdot 10^5 \text{ s} \approx 7250 \text{ min} \approx 121 \text{ h} \approx 5 \text{ days}.$$

β-5. International Space Station. Let us consider the orbit of station as circular, and its average radius was

$$R_0 + h = 6371 \text{ km} + 350 \text{ km} = 6720 \text{ km} \approx 6{,}7 \cdot 10^6 \text{ m},$$

where R_0 – radius of the Earth, and h – the height of the orbit). It is obvious, that the upward jumps are corrections of the orbit by engines, and its slow descent during the lowering is due to loss of energy by the station due to friction in the upper atmosphere. We see that all these parts have an average inclination of approximate 2-2.5 kilometers per month or about 75 meters per day, that is, the change in the altitude of the orbit per day is

$$\Delta h = -75 \text{ m}.$$

To consider correctly all the events, it is better to use energetic approach, calculating the total energy of the station

$$E = \Pi + K = -GMm/(R_0+h) + mV^2/2,$$

and not only kinetic K or potential Π. (Here G is gravitational constant, M is mass of the Earth, V is orbital speed of station.) By the way, note that speed of station increases with lowering of the height (the station is braked, and speed increases!). From conditions of moving on circular orbit

$$GMm/(R_0+h)^2 = mV^2/(R_0+h),$$

Whence

$$GMm/(R_0+h) = mV^2, \; \Pi = -2K, \; E = -K = \Pi/2.$$

Note. The formulae written above are fair for the values of potential energy of gravitational interaction counted from infinity (bodies are removed from each other on infinite distance). Generally speaking, the absolute value of potential energy is not meaningful in physics, only the change of potential energy is important.

Thus,

$$E = -GMm/2(R_0+h).$$

The process of loss of energy by station and simultaneous increase in its speed with lowering of radius of the orbit can be imagined as follows. Let us imagine a quasistationary process with circular orbit all the time, and the work done by forces of resistance of friction

$$A = F\cdot L$$

changes the parameters of this circular orbit. Here F – force of resistance, L – the distance moved. The force $F = \Delta P/\Delta t$ may be found from the following arguments: during each period of time Δt a mass $\mu = \rho\cdot S\cdot V\cdot \Delta t$ of averagely motionless molecules collides with the station (ρ – density of atmosphere at height of orbit of the station). As a result of elastic collisions their speed relative to the station varies from -V to +V, and relative to the Earth – from 0 to 2V. That is, during the period of time Δt the station transmits to the molecules the momentum

$$\Delta P = \mu\cdot 2V = 2V\cdot\rho\cdot S\cdot V\cdot\Delta t = 2\cdot\rho\cdot S\cdot V^2\cdot\Delta t,$$

Whence

$$F = \Delta P/\Delta t = 2\cdot\rho\cdot S\cdot V^2,$$

Thus, during time τ = 24 hours the station moves a distance $L = V\cdot\tau$, and the work of forces of friction (and, accordingly, loss of energy by station) is

$$A = F\cdot L = 2\cdot\rho\cdot S\cdot V^3\cdot\tau,$$

Change in energy

$$\Delta E = -A = -2\cdot\rho\cdot S\cdot V^3\cdot\tau.$$

On the other hand, change in energy of the station in this time is

$$\Delta E = -GMm/2(R_0+h+\Delta h) - \{-GMm/2(R_0+h)\} \approx \Delta h\cdot GMm/2(R_0+h)^2,$$

where Δh – change in height of the orbit (value of Δh is negative!).

$$-2\cdot\rho\cdot S\cdot V^3\cdot\tau = \Delta h\cdot GMm/2(R_0+h)^2,$$

$$\rho = -\Delta h\cdot GMm/\{4(R_0+h)^2\cdot S\cdot V^3\cdot\tau\}$$

and, taking into account that $V^2 = GM/(R_0+h)$,

$$\rho = -m\Delta h / \{4\tau S\cdot(GM)^{1/2}(R_0+h)^{1/2}\}.$$

Numerical calculations: $\rho = 3.12\cdot 10^{-12}$ kg/m^3 $\approx 3\cdot 10^{-12}$ kg/m^3.

So the numerical answer is: $\rho \approx 3\cdot 10^{-12}$ kg/m^3 (more accuracy is not reasonable).

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$$2\pi R_0 \cdot R_0 \cdot 35 \cdot \pi/(180 \cdot 60) \;/\; 4\pi R_0^2 = 35\pi/21600.$$

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but the accuracy even to one star here is obviously excessive, therefore it is necessary to say that about 30 stars become circumpolar due to refraction.

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α-3. Parallax. From the table "Data of some stars" we can find that the distance to Sirius is

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But $3.87 \cdot 10^5$ a.u. is equal to 10^6 semiaxis of orbit of Mercury. So it is reasonable to propose that astronomers of Mercury measure annual parallax in microradians and use this mercurial angular unit for definition of "their parsec". So the mercurial parsec is really par-μ-rad.

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In decimal system $2.1487 = 2 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 8 \cdot 10^{-3} + 7 \cdot 10^{-4}$.
Hence it equals tetradecimal value $K.LMN = K + L \cdot 14^{-1} + M \cdot 14^{-2} + N \cdot 14^{-3} + \ldots$

$$K = [2.1487] = 2, \quad L = [(2.1487-K)/14^{-1}] = 2, \quad M = [(2.1487-K-L \cdot 14^{-1})/14^{-2}] = 1,$$

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That is, the value written in decimal numeral system as 2.1487 will be 2.212 in tetradecimal system.

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Let us consider the second case. L – luminosity of the Sun, r – the radius of the planet, T_W – winter temperature, T_S – summer temperature. For the planet in aphelion (distance to the Sun – R_A) the energy balance is,

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And for the planet in perihelion (distance to the Sun – R_P),

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From these two formulae,

$$R_A \times T_W^2 = R_P \times T_S^2,$$

$$R_A / R_P = (T_S / T_W)^2.$$

$$e = (R_A - R_P) / (R_A + R_P) = (T_S^2 - T_W^2) / (T_S^2 + T_W^2).$$

Numerical calculation (the temperatures expressed in Kelvin) gives us,

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The energy received by a planet from a star is proportional to luminosity (power of radiation) of the star and inversely proportional to square of average distance to a star:

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For the Earth:

$$E_E \sim J_S / R_E^2.$$

For a planet rotating around the white dwarf:

$$E_P \sim J_{WD} / R_P^2.$$

Thus,

$$J_S / R_E^2 = J_{WD} / R_P^2,$$

whence

$$R_P = R_E \times (J_{WD}/J_S)^{1/2}.$$

The ratio of luminosities can be found from the difference of absolute stellar magnitudes,

$$J_{WD}/J_S = 10^{-0.4(Mwd-Ms)}$$

Thus,

$$R_P = R_E \times (10^{-0.4(Mwd-Ms)})^{1/2} = R_E \times 10^{-0.2(Mwd-Ms)}.$$

The absolute stellar magnitude of the Sun M_S should be known by a participant of the Olympiad, or it can be calculated that the value of $m_S = -26.8^m$ of the visible stellar magnitude, from the table and formula for distance of 10 pc written in a.u. (10 pc = 2 062 648 a.u.):

$$M_S = m_S + 5^m \cdot \lg 2062648 = -26.8^m + 31.57^m = 4.77^m.$$

According to general III Kepler law,

$$T^2M/R^3 = \text{const} \qquad \text{therefore} \qquad T_P = T_E(R_P/R_E)^{3/2}(M_{WD}/M_S)^{-1/2}.$$

We know the values T_E and R_P/R_E, then what may be the value of M_S/M_{WD}? Let us remember the Chandrasekhar (*) limit, that is equal to 1.4 M_S – the upper limit for mass of a star that may exist as a white dwarf.

(*) Subrahmanyan Chandrasekhar (Субраманьян Чандраксекар) – American scientist of the Indian origin whose 100 anniversary will be tomorrow, on October 19, 2010.

Thus, the minimal sidereal period is equal

$$T_P = T_E(10^{-0.2(Mwd-Ms)})^{3/2}(1.4)^{-1/2} = T_E \times 10^{-0.3(Mwd-Ms)} \times (1.4)^{-1/2} \approx 0.53 \text{ days}.$$

α-5. Cosmonaut. For estimation we may consider the space station and cosmonaut as pointlike particles. The time $\tau_{½}$ after which these particles collide due to gravitational attraction is approximately equal to half of period of revolution of mass m around mass M on the orbit with the major semiaxis L/2. And the time τ that we should find approximately equals $\tau_{½}$ (neglecting the last part of orbit comparable with the sizes of station and cosmonaut). According to general III Kepler law,

$$4\pi^2/GM = T^2/(L/2)^3, \quad \text{so} \quad T = 2\pi(L/2)^{3/2}(GM)^{-1/2}, \quad \text{and} \quad \tau_{½} = T/2 = \pi(L^3/8GM)^{1/2}$$

$$\tau \approx \tau_{½} = 4.35 \cdot 10^5 \text{ s} \approx 7250 \text{ min} \approx 121 \text{ h} \approx 5 \text{ days}.$$

β-5. International Space Station. Let us consider the orbit of station as circular, and its average radius was

$$R_0 + h = 6371 \text{ km} + 350 \text{ km} = 6720 \text{ km} \approx 6{,}7 \cdot 10^6 \text{ m},$$

where R_0 – radius of the Earth, and h – the height of the orbit). It is obvious, that the upward jumps are corrections of the orbit by engines, and its slow descent during the lowering is due to loss of energy by the station due to friction in the upper atmosphere. We see that all these parts have an average inclination of approximate 2-2.5 kilometers per month or about 75 meters per day, that is, the change in the altitude of the orbit per day is

$$\Delta h = -75 \text{ m}.$$

To consider correctly all the events, it is better to use energetic approach, calculating the total energy of the station

$$E = \Pi + K = -GMm/(R_0+h) + mV^2/2,$$

and not only kinetic К or potential П. (Here G is gravitational constant, M is mass of the Earth, V is orbital speed of station.) By the way, note that speed of station increases with lowering of the height (the station is braked, and speed increases!). From conditions of moving on circular orbit

$$GMm/(R_0+h)^2 = mV^2/(R_0+h),$$

Whence

$$GMm/(R_0+h) = mV^2, \ \Pi = -2K, \ E = -K = \Pi/2.$$

Note. The formulae written above are fair for the values of potential energy of gravitational interaction counted from infinity (bodies are removed from each other on infinite distance). Generally speaking, the absolute value of potential energy is not meaningful in physics, only the change of potential energy is important.

Thus,

$$E = -GMm/2(R_0+h).$$

The process of loss of energy by station and simultaneous increase in its speed with lowering of radius of the orbit can be imagined as follows. Let us imagine a quasistationary process with circular orbit all the time, and the work done by forces of resistance of friction

$$A = F\cdot L$$

changes the parameters of this circular orbit. Here F – force of resistance, L – the distance moved. The force $F = \Delta P/\Delta t$ may be found from the following arguments: during each period of time Δt a mass $\mu = \rho\cdot S\cdot V\cdot\Delta t$ of averagely motionless molecules collides with the station (ρ – density of atmosphere at height of orbit of the station). As a result of elastic collisions their speed relative to the station varies from -V to +V, and relative to the Earth – from 0 to 2V. That is, during the period of time Δt the station transmits to the molecules the momentum

$$\Delta P = \mu\cdot 2V = 2V\cdot\rho\cdot S\cdot V\cdot\Delta t = 2\cdot\rho\cdot S\cdot V^2\cdot\Delta t,$$

Whence

$$F = \Delta P/\Delta t = 2\cdot\rho\cdot S\cdot V^2,$$

Thus, during time τ = 24 hours the station moves a distance $L = V\cdot\tau$, and the work of forces of friction (and, accordingly, loss of energy by station) is

$$A = F\cdot L = 2\cdot\rho\cdot S\cdot V^3\cdot\tau,$$

Change in energy

$$\Delta E = -A = -2\cdot\rho\cdot S\cdot V^3\cdot\tau.$$

On the other hand, change in energy of the station in this time is

$$\Delta E = -GMm/2(R_0+h+\Delta h) - \{-GMm/2(R_0+h)\} \approx \Delta h\cdot GMm/2(R_0+h)^2,$$

where Δh – change in height of the orbit (value of Δh is negative!).

$$-2\cdot\rho\cdot S\cdot V^3\cdot\tau = \Delta h\cdot GMm/2(R_0+h)^2,$$

$$\rho = -\Delta h\cdot GMm/\{4(R_0+h)^2\cdot S\cdot V^3\cdot\tau\}$$

and, taking into account that $V^2 = GM/(R_0+h)$,

$$\rho = -m\Delta h / \{4\tau S\cdot(GM)^{1/2}(R_0+h)^{1/2}\}.$$

Numerical calculations: $\rho = 3.12\cdot10^{-12}$ kg/m^3 $\approx 3\cdot10^{-12}$ kg/m^3.

So the numerical answer is: $\rho \approx 3\cdot10^{-12}$ kg/m^3 (more accuracy is not reasonable).