# GeCAA - Theory Solutions

October 3, 2020

## 1 Astrophotography

An astrophotographer, based on the Equator, uses a good digital camera on a tripod, but with no tracking. The camera has a telescopic lens with focal length of 150 mm and aperture (objective diameter) of 84 mm. The sensor has effective light collecting diameter of 22.5 mm. The photographic target is a star field at the observer's Zenith.

(a) **(2 points)** Calculate the field of view (the angular width of the image) which can be captured on the sensor using this equipment.

(b) **(5 points)** The pixels in the camera's sensor are separated by a distance of 3.23 μm. What is the maximum possible exposure time for a single frame, so that no star trails appear on the exposed image?

(c) **(3 points)** For a better-quality image of the star field, the photographer decides to take multiple shots at the exposure time calculated in b) and then to stack them together. The total time for all these shots is 30 s (ignore any time taken to write data to the memory card) What proportion of the total field of view is possible at the higher signal to noise ratio?

**Solution**

(a) By simple right angle triangle,

$$\text{FOV} = 2 \times \tan^{-1}\left(\frac{\text{sensor width}}{2 \times \text{Focal Length}}\right) = 2 \times \tan^{-1}\left(\frac{22.5}{2 \times 150}\right)$$

$$\alpha \approx 8.58^\circ$$ **2.0**

(b) Number of pixels on the sensor will be given by,

$$N = \frac{\text{Sensor width}}{\text{pixel width}} = \frac{22.5}{3.23 \times 10^{-3}}$$

$$= 6965.94$$ **1.0**

As the number of pixels have to be integer, we take N = 6966 **1.0**

Angular coverage of each pixel will be,

$$\text{Pixel view} = \frac{8.578^\circ}{6966} = 0.001\,23^\circ/\text{pixel}$$

$$\approx 4.43''/\text{pixel}$$ **1.0**

As the stars complete one full circle in 23.9344 hours, **1.0**

student loses this one mark if 15° per hour is used

$$t_1 = 1.23 \times 10^{-3} \frac{23.9344^h}{360} = 0.294\,\mathrm{s}$$
$$\approx 0.3\,\mathrm{s} \qquad \textbf{1.0}$$

(in reality this is probably a factor of 10 smaller than the eye would detect)

(c) In 30 seconds, the sky will move by,

$$\Delta\alpha = \frac{360°}{23.9344 \times 120} = 0.125\,342° \qquad \textbf{1.0}$$
$$\therefore \frac{\alpha - \Delta\alpha}{\alpha} = \frac{8.578 - 0.125342}{8.578}$$
$$= 98.5\% \qquad \textbf{2.0}$$

(or 8.453° is total field of view in high resolution images from the stack) Only penalise 15°/hr once in the whole solution.

## 2 Flat Earth

**(10 points)** A new model of the world is gaining in popularity among some people. These people believe in the "Flat Earth" view of the world, where the Earth is not a spheroid, but rather a circle with radius $R_\oplus$. The central axis of the Earth (normal to the circle passing through its centre C) is passes through the observer's zenith.
This model must at least remain consistent with the observed phenomena, as listed below:

- The value of the solar constant is $S_\odot = 1366\,\mathrm{W/m^2}$.
- The Earth's central axis precesses in a circle with a period 25800 years.
- The radius of the precession circle is 23.5°.

We assume that the Earth is a perfect blackbody radiator and the Sun is sufficiently far away that all sun rays are parallel. Let us also assume that the Sun's current (initial) location is at the zenith.
Determine how many years it will take for the Earth's equilibrium temperature to decrease by 1 °C.

**Solution** Assume the surface area of *one side* of the flat Earth is $A$. Let the angle between the Sun and the flat Earth's center axis be $\theta$, where $\theta$ is initially 0°. As the Sun's rays are parallel, the power delivered to the Earth by the Sun will be $S_\odot A\cos\theta$ at any given point in time.
At equilibrium, this is the energy radiated away via blackbody radiation, so the equilibrium temperature $T$ satisfies

$$S_\odot A\cos\theta = \sigma(2A)T^4$$ **2.0**

, where the factor of 2 comes from the fact that the flat Earth would radiate energy from both sides.
This yields

$$T(\theta) = \sqrt[4]{\frac{S_\odot\cos\theta}{2\sigma}}$$

and we wish to find the value of $\theta_1$ such that $T(\theta_1) = T(0) - \Delta T$.
Thus,

$$\sqrt[4]{\frac{S_\odot\cos\theta_1}{2\sigma}} = \sqrt[4]{\frac{S_\odot}{2\sigma}} - \Delta T$$ **2.0**

$$\cos\theta_1 = \left(1 - \Delta T\sqrt[4]{\frac{2\sigma}{S_\odot}}\right)^4$$ **1.0**

$$= \left(1 - \sqrt[4]{\frac{2\times 5.67\times 10^{-8}}{1366}}\right)^4$$

$$\cos\theta_1 = 0.9880$$ **1.0**

Now, we find the time it takes for the axis to make such an angle with the Sun. On the celestial sphere, let O be the center of precession, Z be the current direction of the axis, and X be direction of the axis when it makes an angle of $\theta_1$ with the sun,

so $\measuredangle ZCX = \theta_1$. If $\epsilon$ is the radius of precession, then $\measuredangle OCZ = \measuredangle OCX = \epsilon$. **1.0**
By the spherical Law of Cosines on angle O of spherical triangle OXZ, we have

$$\begin{aligned}
\cos\theta_1 &= \cos\epsilon\cos\epsilon + \sin\epsilon\sin\epsilon\cos(\sphericalangle O) \\
&= \cos^2\epsilon + \sin^2\epsilon\cos(\sphericalangle O) \\
\sphericalangle O &= \cos^{-1}\left(\frac{\cos\theta_1 - \cos^2\epsilon}{\sin^2\epsilon}\right) && \textbf{1.0} \\
\therefore \Delta t &= \frac{\sphericalangle O}{2\pi} \times P = \frac{P}{2\pi} \times \cos^{-1}\left(\frac{\cos\theta_1 - \cos^2\epsilon}{\sin^2\epsilon}\right) && \textbf{1.0} \\
&= \frac{25800}{2\pi} \times \cos^{-1}\left(\frac{0.9880 - \cos^2 23.5°}{\sin^2 23.5°}\right) \\
&\approx 1606\,\mathrm{yr}
\end{aligned}$$

Thus, the average temperature of the earth will go down by 1 °C in just over **1600 years**. **1.0**

# 3 Mirror

A bored cosmologist comes up with a thought experiment to determine the Hubble constant ($H_0$) for his model of a Steady-State-Universe. In this experiment, a large, fully reflecting flat mirror – carrying several gyroscopes that would maintain its spatial orientation in the same plane – would be placed at a distance $D$ from the Solar System in a region without gravitational influences. From the Earth, a laser beam would be directed towards that region for a long period of time. After a time $T$, the radiation would return and be detected, allowing the determination of the fixed constant $H_0$.

(a) **(7 points)** Find an expression for $H_0$ as a function of $D$, $c$ (speed of light) and $T$. Consider that the separation $S$ between the Solar System and the mirror increases only due to the expansion of the universe according to the law $S = se^{H_0 t}$, where $s$ is the initial separation. You may use $e^x \approx 1 + x$ for $x \ll 1$, if necessary.

(b) **(3 points)** Imagine that such a mirror is located in the vicinity of the star Vega (which also features on the logo of the 1st GeCAA). Vega was the first star outside the Solar System to be photographed and one of the first stars whose parallax ($p = 0.125''$) was accurately measured in 1840 by G. W. von Struve.
Estimate the total duration of this $H_0$ measurement experiment.

**Solution**

(a) Let $t_1$ be the time taken by the light beam from the Solar System to the mirror, let $t_2$ be the time taken by the beam from the mirror to the Solar System and $T$ the total time to go back and forth. As a first order approximation, we will take distance travelled by the photon in each part as an average of the initial and final distance. Therefore, equating the kinematics of the situation, we have:

$$S_1 = \frac{D + De^{H_0 t_1}}{2} = \frac{D\left(1 + e^{H_0 t_1}\right)}{2} = ct_1$$

$$S_2 = \frac{S_1 + S_1 e^{H_0 t_2}}{2} = \frac{S_1\left(1 + e^{H_0 t_2}\right)}{2} = ct_2 \qquad \mathbf{1.0}$$

$$= \frac{D}{4}\left(1 + e^{H_0 t_1}\right)\left(1 + e^{H_0 t_2}\right)$$

$$\approx \frac{D}{4}\left(2 + H_0 t_1\right)\left(2 + H_0 t_2\right)$$

$$\approx \frac{D}{4}\left[4 + 2H_0\left(t_1 + t_2\right)\right]$$

$$S_2 = D\left(1 + \frac{1}{2}H_0 T\right) \qquad \mathbf{1.5}$$

From the first equation, we also find:

$$S_1 = ct_1 = \frac{D\left(1+e^{H_0 t_1}\right)}{2}$$
$$ct_1 = D\left(1+\frac{1}{2}H_0 t_1\right)$$
$$\therefore t_1 = \frac{D}{c-\frac{1}{2}DH_0}$$
$$S_1 = ct_1 = \frac{2Dc}{2c-DH_0} \qquad \textbf{1.5}$$
$$\text{similarly, } S_2 = \frac{2S_1 c}{2c-S_1 H_0}. \qquad \textbf{0.5}$$

Joining the expressions found, we obtain:

$$S_2 = D\left(1+\frac{1}{2}H_0 T\right) = \frac{2S_1 c}{2c-S_1 H_0}$$
$$D\left(1+\frac{1}{2}H_0 T\right) = \frac{\frac{4Dc^2}{2c-DH_0}}{2c-\frac{2Dc}{2c-DH_0}\cdot H_0}$$
$$\left(1+\frac{1}{2}H_0 T\right) = \frac{2c}{2c-DH_0-DH_0} = \frac{c}{c-DH_0}$$
$$c = \left(1+\frac{1}{2}H_0 T\right)(c-DH_0)$$
$$= c - DH_0 + \frac{1}{2}cH_0 T - \frac{1}{2}DH_0^2 T$$
$$0 = \frac{H_0}{2}\left(cT-2D-DH_0 T\right)$$
$$H_0 = \frac{cT-2D}{DT}. \qquad \textbf{2.5}$$

**Alternative solution**

First note that the time taken for the laser beam to travel to the mirror and back again is equal to the time taken for the laser beam to travel a distance of $2D$ (measured at $t=0$) in a straight line. **2.0**

The co-moving distance the beam has to cover is the same in both scenarios. We then equate the distance travelled by the light with the amount by which the space has expanded in that time.

In time $t$, the space has expanded by a factor of $\exp(H_0 t)$, to first order we can linearise this to $1+H_0 t$. This means that the beam, on average, travels through space that's stretched out by a factor of $1+H_0 T/2$. Thus, **2.5**

$$cT = 2D\left(1+\frac{H_0 T}{2}\right)$$

and so

$$H_0 = \frac{cT-2D}{DT}. \qquad \textbf{2.5}$$

**Better approximation (not required in the exam)**

In reality, the light does not travel a distance of $ct$ in this expanding universe. To find the exact expression, consider a time differential $\mathrm{d}t$ time $t$ after the beam was emitted. In this small time interval, the beam travels a distance of $c\mathrm{d}t$. Because the space is stretched out, the travelled distance corresponds to a smaller segment of space at $t = 0$, smaller by a factor of $\exp(H_0 t)$. The distance spanned at $t = 0$ is then

$$\mathrm{d}r = \exp(-H_0 t)c\mathrm{d}t.$$

We integrate this from $t = 0$ to $t = T$:

$$\begin{aligned}\int_0^{2D} \mathrm{d}r &= c\int_0^T \exp(-H_0 t)\mathrm{d}t,\\ 2D &= \frac{c}{H_0}\left(1 - \exp(-H_0 T)\right).\end{aligned}$$

The result so far is accurate within the constraints of the model, but it's not analytically solvable for $H_0$. To get an estimate, we can approximate the right hand side to

$$2D = \frac{c}{H_0}\left(1 - 1 + H_0 T - \frac{(H_0 T)^2}{2}\right) = cT - \frac{cH_0 T^2}{2}.$$

Expressing $H_0$, we get

$$H_0 = \frac{2}{cT^2}\left(cT - 2D\right).$$

The difference between this answer and the initial estimate is $2D/cT$ which is almost unitary.

(b) From the $H_0$ expression found in the previous item, we find the travel time

$$T = \frac{2D}{c - DH_0} \qquad \mathbf{1.0}$$

$$= \left(\frac{c}{2D} - \frac{H_0}{2}\right)^{-1} \approx \left(\frac{c}{2D}\right)^{-1} = \frac{2D}{c} \qquad \mathbf{1.0}$$

$$T = \frac{2 \times 8 \times 3.086 \times 10^{16}}{3 \times 10^8}$$

$$= 1.65 \times 10^9\,\mathrm{s}$$

$$T \approx 52.2\,\mathrm{yr}. \qquad \mathbf{1.0}$$

# 4 Light Curves

The light curve A shown below, shows a fictional edge-on eclipsing binary system containing stars X (radius $r_X$, luminosity $L_X$) and Y (radius $r_Y$, Luminosity $L_Y$) . Assume that star X is brighter, but star Y is hotter.

(a) **(1 point)** Which of the two stars is likely to be on the main sequence? (Write "X" or "Y")

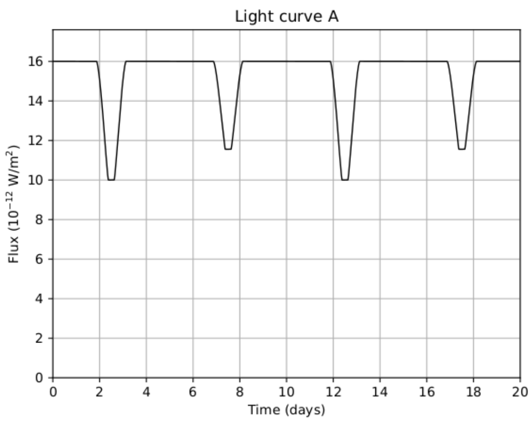
(b) Based on light curve A, estimate:

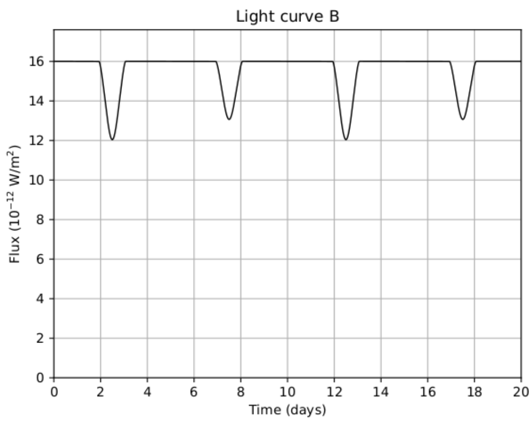
(I) **(2 points)** $\frac{r_X}{r_Y}$, the ratio of the radii of the two stars.

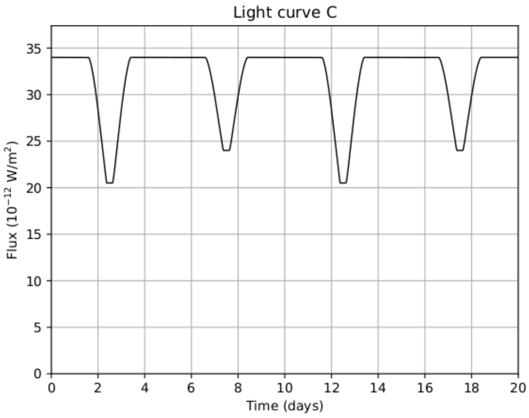
(II) **(2 points)** $\frac{L_X}{L_Y}$, the ratio of the Luminosity of the two stars.

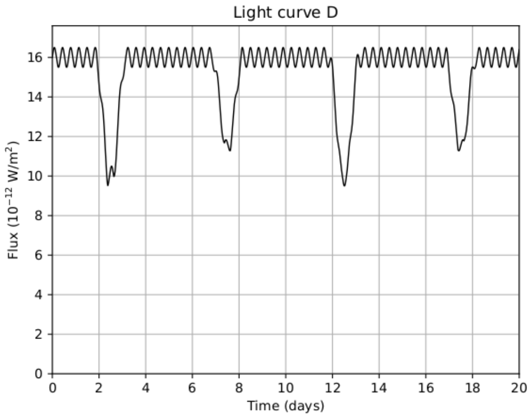
(c) **(15 points)** For light curves B to F, in each case only one parameter of the binary system has been changed from the case in light curve A. For each case, choose the description from the following list that best corresponds to the change (Write the appropriate roman numeral in the answer sheet).

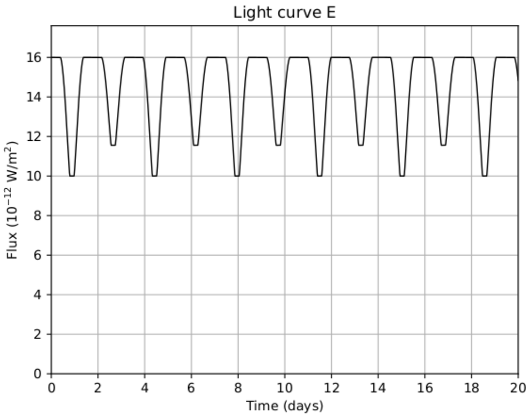
(i) Star X increased in size.
(ii) Star X increased in luminosity.
(iii) Star X decreased in size.
(iv) Star X decreased in luminosity.
(v) Star Y increased in size.
(vi) Star Y increased in luminosity.
(vii) Star Y decreased in size.
(viii) Star Y decreased in luminosity.
(ix) Star X is a variable star.
(x) Star Y is a variable star.
(xi) The inclination of the system relative to the Earth has changed.
(xii) The distance of the system from the Earth has decreased.
(xiii) The distance of the system from the Earth has increased.
(xiv) The orbital period of the system increased.
(xv) The orbital period of the system decreased.

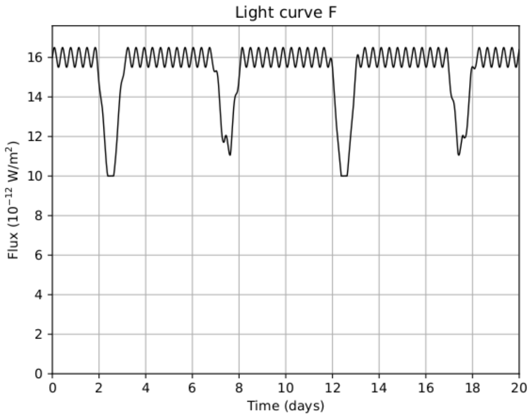





**Solution**

(a) As "**Y**" is smaller, yet hotter, it must be on the main sequence. **1.0**

(b) (I) For this we have to measure the total duration of eclipse and compare it with bottom flat part (total eclipse phase) in the lightcurve.
Total eclipse time should be $(1.25 \pm 0.10)$ days, flat period of eclipse $(0.25 \pm 0.05)$ days. **1.0**

$$\frac{r_X + r_Y}{r_X - r_Y} = \frac{1.25 \pm 0.10}{0.25 \pm 0.05} = 5.0 \pm 1.5$$
$$\therefore \frac{r_X}{r_Y} = \frac{5+1}{5-1} = 1.5 \pm 0.2$$ **1.0**

(II) Because star Y is hotter, star Y will be brighter than a part of star X with the same area. Thus, the deeper dip in the light curve is due to star Y passing behind star X. Thus, the flux from star X must be $10 \times 10^{-12}\,\mathrm{W/m^2}$. As the combined flux is $16 \times 10^{-12}\,\mathrm{W/m^2}$, the flux of star Y is $6 \times 10^{-12}\,\mathrm{W/m^2}$. Thus

$$\frac{L_X}{L_Y} = \frac{10 \times 10^{-12}}{6 \times 10^{-12}} = 1.67$$

**2.0**

(c)

- **Light Curve B:** We see that the dips in brightness both become shallower and have become rounded at the bottom, while the maximum brightness is still $16 \times 10^{-12}\,\mathrm{W/m^2}$. If radius of one of the stars had changed (rounded bottom), then we would expect that total eclipse period to also change. But that has not happened. Brightening of star X can explain shallower dip but not rounded trough. This means that the stars no longer fully eclipse each other. Hence,
  **(cxi): The inclination of the system relative to the Earth has changed.** **4.0**
- **Light Curve C:** The y-axis shows that the overall flux has increased. Since flux of both the stars has gone up by the same factor and there is no substantial change in the profile of eclipse,
  **(cxii): The distance of the system from the Earth has decreased.** **4.0**
- **Light Curve D:** The regular fluctuations in brightness indicate the one components of the system is a variable star. Since the fluctuations persist through all the dips in flux, this indicates that the variable star is never completely eclipsed. Therefore,
  **(cix): Star X is a variable star.** **3.0**
- **Light Curve E:** The overall flux and dips in the flux are the same size, but occur much more frequently. This means that
  **(cxv): The orbital period of the system decreased.** **2.0**
- **Light Curve F:** The regular fluctuations again indicate that one of the components of the system is a variable star. During the primary eclipse, light curve is perfectly flat. This can only occur if,
  **(cx): star Y is a variable star.** **2.0**

# 5 HII region

Luminous Blue Variable (LBV) are massive, unstable, supergiant stars that can undergo episodes of very strong mass loss, due to an instability in their atmospheres. After such an event, a dense nebula is formed around the star. LBV are also very hot stars and produce a large amount of high-energy photons that are able to ionise hydrogen atoms ($E_{ph} > h\nu_0 = 13.6\,\mathrm{eV}$) creating a roughly spherical region of ionized hydrogen (HII region).

In this problem, we consider a static, homogeneous, pure hydrogen nebula with a concentration of $n_H = 10^8\,\mathrm{m^{-3}}$ and temperature $T_{HII} = 10^4\,\mathrm{K}$, ionized by photons from a single LBV star with a stable rate of ionizing photons $Q = 10^{49}\,\mathrm{ph/s}$. Assume that each photon can ionise only one hydrogen atom. At a particular location within an HII region, the rate of photoionization is balanced by the rate of recombination per unit volume. This sets the radius of the fully ionized region and this region is called the Stromgren sphere with the radius $R_S$.

The total number of recombinations per volume is proportional to the concentration of protons $n_p$, the concentration of electrons $n_e$ and the recombination coefficient for hydrogen $\alpha(T_{HII}) = 10^{-19}\,\mathrm{m^3\,s^{-1}}$. For simplification, ignore the fact that the process of recombination can also release ionising photons.

(a) **(5 points)** Derive an algebraic expression for the radius of the Stromgren sphere and calculate its value for the given parameters. Express your answer in units of parsecs (pc).

(b) **(3 points)** The photoionization cross-section of H-atoms in the ground state encountering photons with frequency $\nu_0$ is equal to

$$\sigma \approx 10^{-21}\,\mathrm{m^2}$$

Calculate the mean-free path $l_{\nu_0}$ of an ionising photon. Compare $l_{\nu_0}$ to $R_S$ to determine if this ionized nebulae is sharp-edged or not? (answer "YES" or "NO")

(c) **(4 points)** On what timescale (in years) do you expect the Stromgren sphere to form?

(d) **(4 points)** Radiation from an ionized hydrogen cloud (HII region) is often called free-free emission because it is produced by free electrons scattering off the ions without being captured: the electrons are free before the interaction and remain free afterwards. In this process, the electron retains most of its pre-scattering energy.
An electron, while passing by a much more massive singly ionized hydrogen atom, produces a radio photon of $\nu = 10\,\mathrm{GHz}$. Calculate the mean electron thermal energy in the HII region, for the given temperature of the Stromgren sphere. Is this an example of free-free emission? (answer "YES" or "NO")

(e) **(4 points)** Since the HII region is in local thermodynamic equilibrium, one can calculate the absorption coefficient that is proportional to the optical depth $\tau_\nu \propto \nu^{-2.1}$ and it turns out that at the sufficiently high radio frequencies, the hot plasma is nearly transparent and hence, $\tau_\nu \ll 1$.
The flux density of photons has power-law spectra of the form $S_\nu \propto \nu^\beta$.
Find $\beta$ for the radio frequencies.

**Solution**

(a) As the number of H-atoms undergoing ionization and recomination are balanced at $R_S$, each photon can ionize exactly one hydrogen atom and each neutral hydrogen has exactly one proton and one electron,

$$\begin{aligned} n_{\text{recomb}} &= n_{HII} = Q \\ \text{and } n_e &= n_p = n_H && \textbf{2.0} \\ n_{\text{recomb}} &= \alpha n_p n_e V_S \\ Q &= \alpha n_H^2 \frac{4\pi}{3} R_S^3 && \textbf{1.0} \\ \therefore R_S &= \sqrt[3]{\frac{3Q}{4\pi\alpha n_H^2}} && \textbf{1.0} \\ &= \sqrt[3]{\frac{3\times 10^{49}}{4\pi \times 10^{-19} \times (10^8)^2}} \\ &= 1.3\times 10^{17}\,\text{m} \\ \therefore R_S &\approx 4\,\text{pc} && \textbf{1.0} \end{aligned}$$

(b) Per one unit length distance, a typical photon will encounter $\sigma n_H$ H-atoms. Thus, the mean free path will be,

$$\begin{aligned} l_{\nu_0} &= \frac{1}{\sigma n_H} = \frac{1}{\sigma 10^{-21}\times 10^8} \\ l_{\nu_0} &= 10^{13}\,\text{m} && \textbf{2.0} \\ \therefore l_{\nu_0} &\approx 10^{-4} R_S \ll R_S \end{aligned}$$

Thus, the boundary layer of the sphere is very thin as compared to its total size. Hence,
"YES" this ionized nebula is very sharp-edged. **1.0**

(c) For Stromgren sphere to form, all H-atoms ($N$) inside $R_S$ need to be ionized. Thus, time $t_S$ required will be

$$\begin{aligned} N &= V_S n_H = \frac{4\pi}{3} R_S^3 n_H && \textbf{1.0} \\ t_S &= \frac{N}{Q} = \frac{4\pi R_S^3 n_H}{3Q} && \textbf{1.0} \\ &= \frac{1}{\alpha n_H} = \frac{1}{10^{-19}\times 10^8} \\ &= 10^{11}\,\text{s} \\ &\approx 3000\,\text{yr} && \textbf{2.0} \end{aligned}$$

For typical nebular densities, the main-sequence lifetime of LBV stars is much longer than this ionization time, and hence our assumption of a stationary system is justified.

(d) The mean electron thermal energy in a plasma of temperature $T_e = 10^4\,\mathrm{K}$ is

$$E_e \approx k_B T_e \approx 1.4 \times 10^{-19}\,\mathrm{J} \approx 0.9\,\mathrm{eV}$$ **2.0**

The energy of a photon is

$$E_\gamma = h\nu \approx 6.6 \times 10^{-24}\,\mathrm{J} \approx 4 \times 10^{-5}\,\mathrm{eV}$$ **1.0**

As the energy of the radio photon is much much smaller than that of the electron, the answer is "Yes". **1.0**

(e) As the plasma is nearly transparent, using the Rayleigh-Jeans law,

$$B_\nu = \frac{2k_B T_e \nu^2}{c^2}$$ **1.0**

$$S_\nu \propto B_\nu \tau_\nu = \frac{2k_B T_e \nu^2}{c^2}\tau_\nu$$

$$S_\nu \propto \nu^2 \cdot \nu^{-2.1}$$

$$\therefore S_\nu \propto \nu^{-0.1}$$ **3.0**

# 6 Occultation of a X-ray Source

Consider a satellite observing x-ray sources, while orbiting the Earth in the equatorial plane with orbital radius $r$, and orbital time period $P$. Let us assume that this satellite is pointed to one fixed direction in space for a given length of time. Take the radius of the earth as $R$.

When the satellite moves 'behind' the earth, naturally, the x-ray source is 'occulted' and the measured x-ray flux from the source drops to zero. However, due to Earth's atmosphere, this drop is gradual. If the line of sight of the source passes through the atmosphere, the attenuation depends on the air-mass (i.e. length of air column) along the line of sight.

(a) **(1 point)** Let us assume that pointing towards a fixed source at 0° declination. We consider that the source is occulted when 50% of the light coming from the source gets attenuated due to the atmosphere. Let us say that this happens when the minimum height of the line of sight from the surface of the Earth is $h$.

If $\theta_0$ is the angle between the direction to the source and the direction to the Earth, as measured from the spacecraft, find an expression for $\theta_0$.

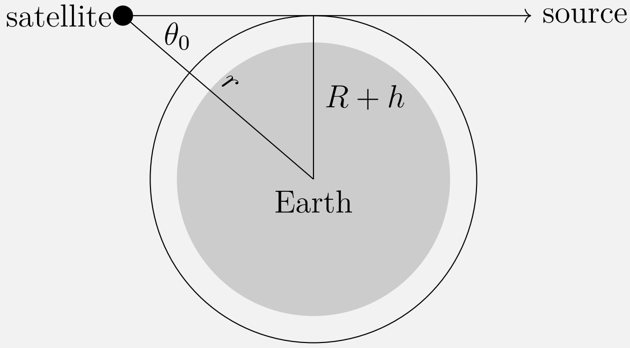
(b) **(4 points)** The time duration $\Delta t$ between the source getting attenuated from 90% of pre-occultation flux to 10% is defined as the 'occultation time' for the source. Assume the flux attenuates to 90% when the minimum height of the line of sight $(h + 0.5\Delta h)$ and similarly the flux attenuates to 10% at $(h - 0.5\Delta h)$, where $\Delta h \ll R$.
Find the expression for $\Delta t$ in terms of $r$, $P$, $\Delta h$ and $\theta_0$.

(c) **(15 points)** If the satellite was pointing towards a source at declination $\beta$ instead ($\beta$ not too large), what will be the expression for $\Delta t$?

**Note:**
If the satellite was not in the equatorial plane, then the problem could have been simply rephrased by assuming the satellite's orbital plane to be the equatorial plane. In that case, $\beta$ becomes 'relative declination'.

**Solution**

(a) The solution for first part is obvious from the figure below.



$$\sin\theta_0 = \left(\frac{R+h}{r}\right)$$

$$\therefore \theta_0 = \sin^{-1}\left(\frac{R+h}{r}\right)$$ **1.0**

(b) Let us say the satelite has to move in the orbit by $\Delta\theta/2$ for $0.5\Delta h/2$ change in height.

$$\sin\left(\theta_0 \pm \frac{\Delta\theta}{2}\right) = \left(\frac{R+h \pm 0.5\Delta h}{r}\right)$$ **1.0**

$$\sin\left(\theta_0 + \frac{\Delta\theta}{2}\right) - \sin\left(\theta_0 - \frac{\Delta\theta}{2}\right) = \left(\frac{\Delta h}{r}\right)$$

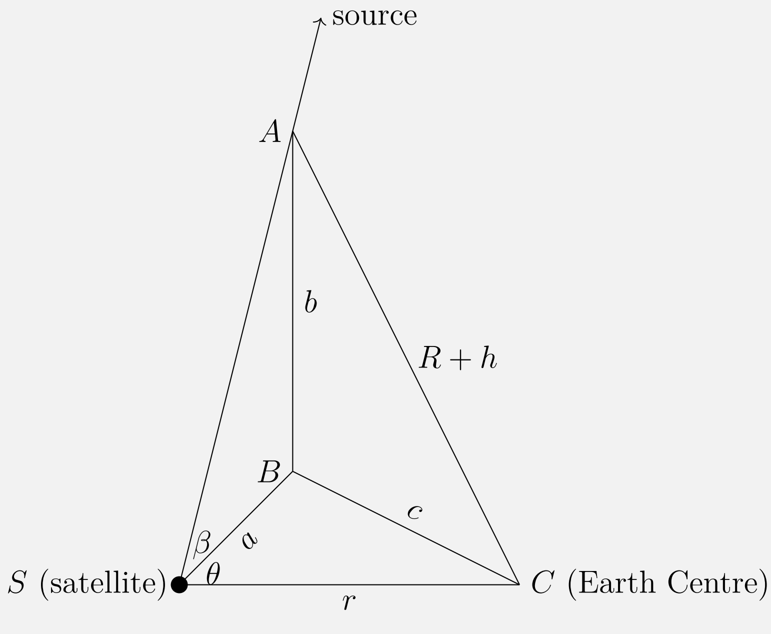
$$\therefore 2\cos\theta_0 \sin\left(\frac{\Delta\theta}{2}\right) = \left(\frac{\Delta h}{r}\right)$$

$$2 \times \frac{\Delta\theta}{2}\cos\theta_0 = \left(\frac{\Delta h}{r}\right)$$ **1.0**

$$\Delta\theta = \left(\frac{\Delta h}{r\cos\theta_0}\right) = \frac{2\pi\Delta t}{P}$$ **1.0**

$$\therefore \Delta t = \left(\frac{P\Delta h}{2\pi r\cos\theta_0}\right)$$ **1.0**

(c) Let $\theta$ be the new angle for 50% attenuation. In the diagram below, The line of sight from the satelite (S) to the source is at minimum distance from the centre of the earth (C) at point A. The points S, C and B are in the equatorial plane with line BA normal to the equatorial plane. One also realises that the line BS is normal to the plane defined by A, B and C. **3.0**



$$a = r\cos\theta$$
$$c = r\sin\theta$$
$$b = a\tan\beta$$
$$(R+h)^2 = b^2 + c^2$$
$$= (r\cos\theta\tan\beta)^2 + (r\sin\theta)^2 \qquad \textbf{3.0}$$
$$\therefore \frac{(R+h)^2}{r^2} = \cos^2\theta\tan^2\beta + \sin^2\theta$$
$$= (1-\sin^2\theta)\tan^2\beta + \sin^2\theta$$
$$= \tan^2\beta + \sin^2\theta(1-\tan^2\beta)$$
$$\therefore \sin^2\theta = \left(\frac{(R+h)^2}{r^2} - \tan^2\beta\right)\left(\frac{1}{(1-\tan^2\beta)}\right) \qquad \textbf{1.0}$$
$$\text{Let us say} \quad \tan^2\beta = k$$
$$\therefore \sin\theta = \sqrt{\left(\frac{(R+h)^2}{r^2} - k\right)\left(\frac{1}{(1-k)}\right)}$$

Using similar analysis as the previous part,

$$\Delta\theta\cos\theta = \sqrt{\frac{1}{(1-k)}}\left[\sqrt{\left(\frac{(R+h+\frac{\Delta h}{2})^2}{r^2} - k\right)} - \sqrt{\left(\frac{(R+h-\frac{\Delta h}{2})^2}{r^2} - k\right)}\right] \qquad \textbf{2.0}$$
$$= \sqrt{\frac{1}{(1-k)}}\left(\frac{\Delta h(R+h)}{r^2}\right)\left(\frac{(R+h)^2}{r^2} - k\right)^{-0.5} \qquad \textbf{2.0}$$
$$= \sqrt{\frac{1}{(1-k)}\left(\frac{(R+h)^2}{r^2} - k\right)}\left(\frac{\Delta h}{r} \times \frac{(R+h)}{r}\right)\left(\left(\frac{(R+h)}{r}\right)^2 - k\right)^{-1} \qquad \textbf{2.0}$$
$$= \sin\theta\left(\frac{\Delta h\sin\theta_0}{r}\right)\left(\sin^2\theta_0 - \tan^2\beta\right)^{-1}$$
$$\therefore \Delta t = \left(\frac{P\Delta h\tan\theta}{2\pi r}\right)\left(\frac{\sin\theta_0}{\sin^2\theta_0 - \tan^2\beta}\right) \qquad \textbf{1.0}$$

# 7 Radiant of a Meteor Shower

A stargazer in Chiayi, Chinese Taipei (23.5°N, 120.4°E, GMT+8) saw two meteors streaking through the sky at 21:00 (Chinese Taipei time) on 25$^{\text{th}}$ September 2020. One of the meteors appeared at horizon exactly due west and streaked to a point at 15° altitude directly above the northern horizon. The second meteor originated at an altitude of 23.5° and an azimuth of 210° and ended at an altitude of 75° and an azimuth of 255°.

(a) **(6 points)** What is the Local Sidereal Time (LST) at the time of observation?

(b) **(16 points)** Find the alt-az coordinates of the apparent radiant of the two meteors.

(c) **(6 points)** Find the equatorial coordinates of the apparent radiant.

(d) **(2 points)** Which of the following constellations is closest to the radiant?
Crux / Dorado / Pavo / Tucana / Triangulum Australes
(choose one and write the same name in the answer box)

**Notes:**

- Azimuths are measured from the North (0°) towards the East.
- The Greenwich Sidereal Time (GST) at 00:00 UT on 1$^{\text{st}}$ January 2020 is $6^{\text{h}}$ $40^{\text{m}}$ $30^{\text{s}}$.

**Solution**

(a) Step 1: Determine the GST at the time of observation.

Let us denote the GST on at 00:00 UT on 1$^{\text{st}}$ January 2020 as $\text{GST}_0$.
Total 268 days have elapsed since start of the year till $0^{\text{h}}$ UT on 25 September. **1.0**

At 21:00 for GMT+8 timezone, UT will be $21^{\text{h}}$-$8^{\text{h}}$ = $13^{\text{h}}$
Hence,

$$\begin{aligned} GST &= GST_0 + \Delta t \\ &= 6^h 40^m 30^s + \frac{268 \times 24}{365.2422} + \frac{13 \times 24}{23.9344} \\ &\approx 6^h 40^m 30^s + 17^h 36^m 36^s + 13^h 2^m 8^s \\ &= 37^h 19^m 14^s = \mathbf{13^h 19^m 14^s} \end{aligned}$$

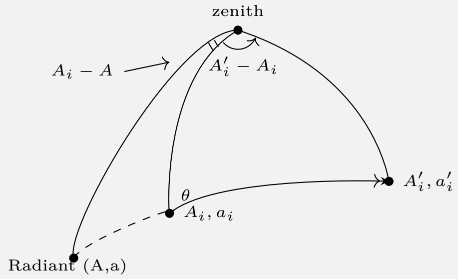
**3.0**

As longitude of observer is 120.4°, the LST of observation is,

$$\begin{aligned} LST &= GST + 24^h \times \frac{120.4°}{360°} \\ &= 13^h 19^m 14^s + 8^h 1^m 36^s \\ &= \mathbf{21^h 20^m 50^s} \end{aligned}$$

**2.0**

(b) For horizontal coordinates




**1.0**

The spherical triangle has the radiant, initial position and final position of the $i$-th meteor to be $(A, a), (A_i, a_i)$ and $(A_i', a_i')$, where $a$ and $A$ denotes the altitude and azimuth respectively.
Using the four-parts (cotangent) equation on the left and right triangles, we have, for both meteors:

$$\cos(90 - a_i)\cos(A_i' - A_i) = \sin(90 - a_i)\cot(90 - a_i') - \sin(A_i' - A_i)\cot\theta$$
$$\therefore \sin a_i \cos(A_i' - A_i) = \cos a_i \tan a_i' - \sin(A_i' - A_i)\cot\theta$$

Similarly,

$$\begin{aligned}\cos(90 - a_i)\cos(A_i - A) &= \sin(90 - a_i)\cot(90 - a) \\ &\quad - \sin(A_i - A)\cot(180 - \theta) \\ \therefore \sin a_i \cos(A_i - A) &= \cos a_i \tan a + \sin(A_i - A)\cot\theta\end{aligned}$$

**2.0**

We can then eliminate $\cot\theta$ to yield:

$$\frac{\sin a_i \cos(A_i - A)}{\sin(A_i - A)} - \frac{\cos a_i \tan a}{\sin(A_i - A)} = \frac{\cos a_i \tan a_i'}{\sin(A_i' - A_i)} - \frac{\sin a_i \cos(A_i' - A_i)}{\sin(A_i' - A_i)} = \cot\theta$$

multiplying the whole equation by $\dfrac{\sin(A_i' - A_i)\sin(A_i - A)}{\cos a_i}$

$$\begin{aligned}\tan a_i \cos(A_i - A)\sin(A_i' - A) &\qquad \tan a_i' \sin(A_i - A) \\ - \tan a \sin(A_i' - A) &= - \tan a_i \cos(A_i' - A_i)\sin(A_i - A) \\ \tan a_i \sin[(A_i' - A_i) + (A_i - A)] &= \tan a_i' \sin(A_i - A) + \tan a \sin(A_i' - A_i) \\ \tan a_i \sin(A_i' - A) &= \tan a_i' \sin(A_i - A) + \tan a \sin(A_i' - A_i)\end{aligned}$$

**3.0**

Plugging in $i = 1, 2$, and eliminating $\tan a$ by division, we have

$$\frac{\tan a_1 \sin(A_1' - A) - \tan a_1' \sin(A_1 - A)}{\tan a_2 \sin(A_2' - A) - \tan a_2' \sin(A_2 - A)} = \frac{\sin(A_1' - A_1)}{\sin(A_2' - A_2)} \stackrel{\text{def}}{=} k$$

where $k$ is the RHS defined above (which is a known value). Expanding the equation above and dividing by $\cos A$, gathering terms of $\tan A$, we get

$$\tan A = \frac{\tan a_1 \sin A_1' - \tan a_1' \sin A_1 - k \tan a_2 \sin A_2' + k \tan a_2' \sin A_2}{\tan a_1 \cos A_1' - \tan a_1' \cos A_1 - k \tan a_2 \cos A_2' + k \tan a_2' \cos A_2}$$

**2.0**

$$\text{and } \tan a = \frac{\tan a_1 \sin(A_1' - A) - \tan a_1' \sin(A_1 - A)}{\sin(A_1' - A_1)}$$

Plugging in the respective values from the question,

$$\begin{aligned} A_1 &= 0^\circ & a_1 &= 15^\circ \\ A_1' &= 90^\circ & a_1' &= 0^\circ \\ A_2 &= 210^\circ & a_2 &= 23.5^\circ \\ A_2' &= 255^\circ & a_2' &= 75^\circ \end{aligned}$$

We get

$$\begin{aligned} k &= \frac{\sin(A_1' - A_1)}{\sin(A_2' - A_2)} = \frac{\sin 90^\circ}{\sin 15^\circ} \\ k &= 1.414 \\ \tan a_1 \sin A_1' &= 0.268 \\ \tan a_1' \sin A_1 &= 0 \\ k \tan a_2 \sin A_2' &= -0.594 \\ k \tan a_2' \sin A_2 &= -2.639 \\ \tan a_1 \cos A_1' &= 0 \\ \tan a_1' \cos A_1 &= 0 \\ k \tan a_2 \cos A_2' &= -0.159 \\ k \tan a_2' \cos A_2 &= -4.571 \\ \therefore \tan A &= \frac{0.268 - 0 + 0.594 - 2.639}{0 - 0 + 0.159 - 4.571} = 0.403 \\ \therefore A &= 21.94^\circ \\ \text{OR } A &= 201.94^\circ \end{aligned}$$ **4.0**

However, the student should realize that for $A = 21.94^\circ$, the radiant is in the path of the first meteor, which is unrealistic. Hence, it is more likely that the azimuth takes the second value. **1.0**

Plugging in for the altitude, we find the coordinate of the radiant to be

$$\begin{aligned} \tan a_1 \sin(A_1' - A) &= -0.249 \\ \tan a_1' \sin(A_1 - A) &= 0 \\ \sin(A_1' - A_1) &= 1 \\ \therefore \tan a &= (-0.2485 - 0)/1 = -0.2485 \\ a &= -13.96^\circ \end{aligned}$$ **2.0**

As the meteor track origin and end points are rounded to integer degrees, the horizontal coordinates of the radiant will be

$$(\mathbf{A}, \mathbf{a}) = (202^\circ, -14^\circ)$$ **1.0**

**Alternative solution**

In the vector representation, let the starting and final locations of the meteroids be $\vec{u}_i$ and $\vec{v}_i$ respectively. For the subsequent calculations, the vectors are most conveniently represented in Cartesian coordinates. Letting the $x$-axis point east, $y$-axis north, and $z$-axis towards the zenith, we have

$$\vec{u} = (u_x, u_y, u_z) = (\cos a \sin A, \cos a \cos A, \sin a) \tag{1}$$

for a generic vector $\vec{u}$ with azimuth $A$ and altitude $a$. We can therefore compute

$$\begin{aligned}
\vec{u}_1 &= (\cos 15° \sin 0°, \cos 15° \cos 0°, \sin 15°) = (0, 0.9659, 0.2588)\,, \\
\vec{u}_2 &= (\cos 23.5° \sin 210°, \cos 23.5° \cos 210°, \sin 23.5°) = (-0.4585, -0.7942, 0.3988)\,, \\
\vec{v}_1 &= (\cos 0° \sin 90°, \cos 0° \cos 90°, \sin 0°) = (1, 0, 0)\,, \\
\vec{v}_2 &= (\cos 75° \sin 255°, \cos 75° \cos 255°, \sin 75°) = (-0.25, -0.0670, 0.9659)\,.
\end{aligned}$$ **3.0**

The meteors move along great arcs, and the two intersections of the great arcs correspond to the two possible locations of the radiant. A great arc is uniquely defined by its normal vector, denote it with $\vec{n}_i$. The normal vector is conveniently calculated through the cross product of the starting and final position, $\vec{n}_i = \vec{u}_i \times \vec{v}_i$. The cross product can be calculated via the determinant of the following matrix

$$\vec{u}_i \times \vec{v}_i = \begin{vmatrix} i & j & k \\ u_{ix} & u_{iy} & u_{iz} \\ v_{ix} & v_{iy} & v_{iz} \end{vmatrix} = (u_{iy}v_{iz} - u_{iz}v_{iy}, u_{iz}v_{ix} - u_{ix}v_{iz}, u_{ix}v_{iy} - u_{iy}v_{ix}).$$ **4.0**

This yields

$$\begin{aligned}
\vec{n}_1 &= (0, 0.2588, -0.9659)\,, \\
\vec{n}_2 &= (-0.7404, 0.3432, -0.1678)\,.
\end{aligned}$$ **1.0**

Every point on the great circle $i$ is perpendicular to $\vec{n}_i$. This means that the intersections of the two great circles are perpendicular to both $\vec{n}_1$ and $\vec{n}_2$. The vector parallel to the intersections can therefore be calculated by the cross product of $\vec{n}_1$ and $\vec{n}_2$. Let the vector corresponding to the intersection be $\vec{m}$. Then

$$\vec{m} = \vec{n}_1 \times \vec{n}_2$$ **3.0**

$$= (0.3626, 0.9002, 0.2412)\,.$$ **1.0**

Now all that's left is to calculate the azimuth and altitude corresponding to $\vec{m}$. Note that $-\vec{m}$ is also an intersection. The equations for the altitude and azimuth can be reverse-engineered from equation (1):

$$\begin{aligned}
a &= \arctan\left(\frac{z}{\sqrt{x^2 + y^2}}\right), \\
A &= \arctan\left(\frac{x}{y}\right).
\end{aligned}$$ **1.0**

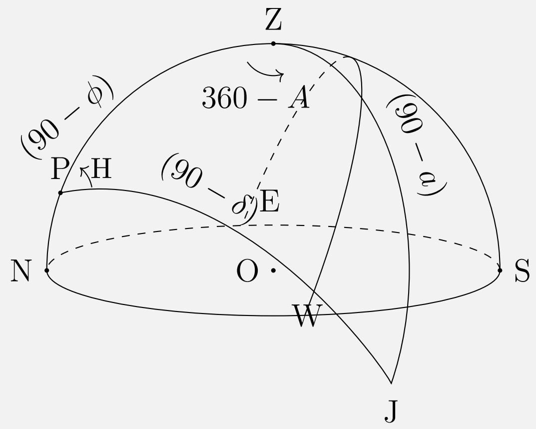
One has to take care that the arctans are signed. This gives the final values of the intersection points to be

$$(A, a) = (22°, 14°) \text{ and } (A, a) = (202°, -14°).$$ **2.0**

As in the first solution, the $A = 22°$ solution can be eliminated due to it being on the path of the first meteor This gives the final answer of

$$(\mathbf{A}, \mathbf{a}) = (202°, -14°)$$ **1.0**

(c) Now, it is easy to convert the horizontal coordinates of the radiant to equatorial coordinates.




**1.0**

Using the cosine rule:

$$\begin{aligned}
\sin\delta &= \sin\phi\sin a + \cos\phi\cos a\cos A \\
&= \sin(23.5°)\sin(-13.4°) + \cos(23.5°)\cos(-13.96°)\cos(201.94°) \\
&= -0.9217 \\
\therefore \delta &= -67.2°
\end{aligned}$$ **2.0**

$$\begin{aligned}
\cos H &= \frac{\sin a - \sin\delta\sin\phi}{\cos\phi\cos\delta} \\
&= \frac{\sin(-13.4°) - \sin(-67.2°)\sin(23.5°)}{\cos(23.5°)\cos(-67.2°)} = 0.3551 \\
\therefore H &= 4^h36^m47^s
\end{aligned}$$ **1.5**

From the relationship between the LST and the Hour Angle ($H = LST - \alpha$), we get the equatorial coordinates of the radiant to be

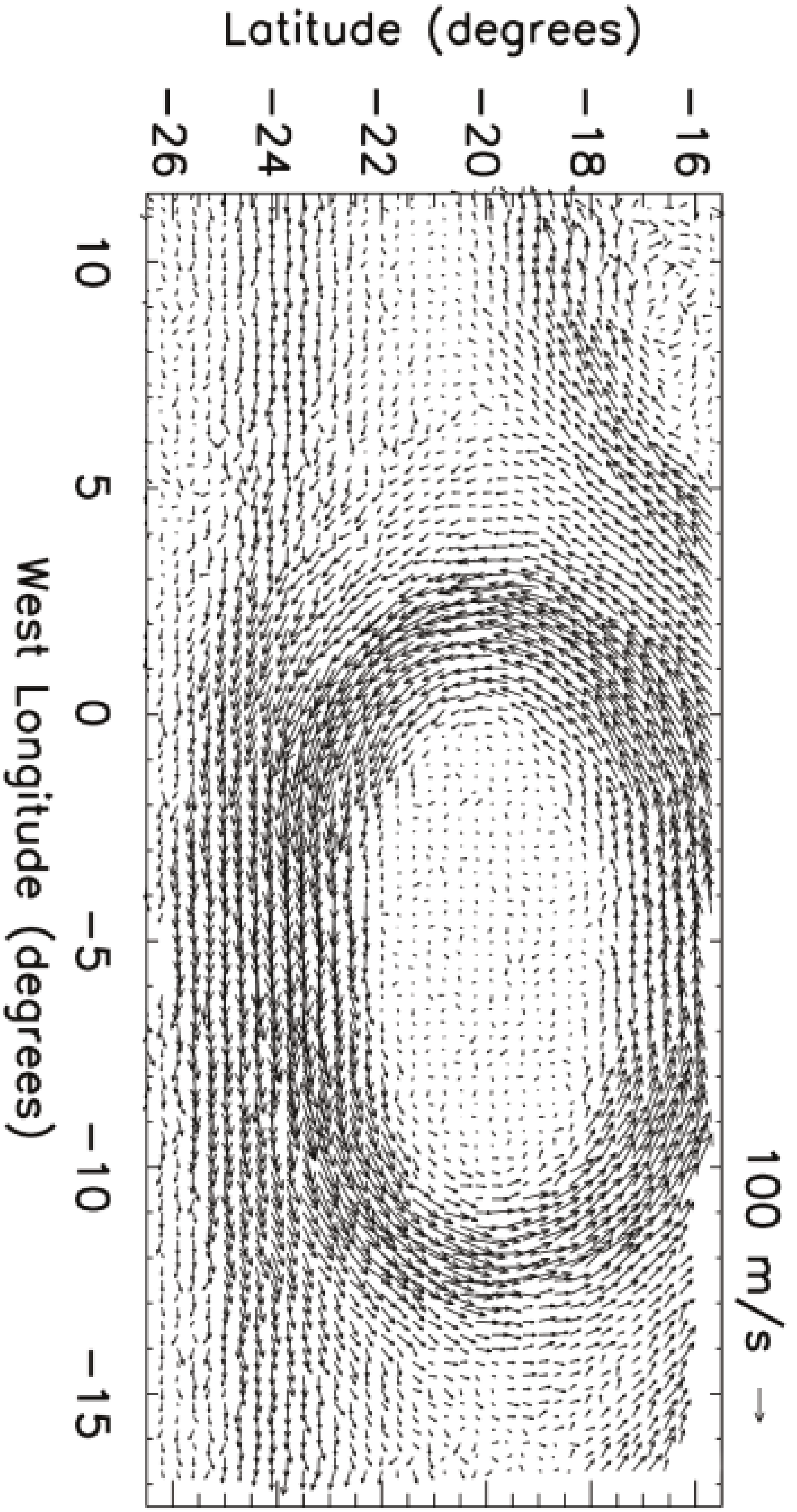
$$\begin{aligned}
\alpha &= 21^h20^m50^s - 4^h36^m47^s \\
&= 16^h44^m3^s
\end{aligned}$$ **1.5**

Thus, the equatorial coordinates of the radiant are,

$$(\alpha, \delta) = (16^h44^m, -67.2°)$$

(d) Triangulum Australes **2.0**

## 8 Jupiter's Great Red Spot




In the following problem the fluid mechanics of Jupiter's Great Red Spot (GRS) is studied based on the velocity field data. The diagram on the next page shows a map of relative velocity for GRS and the surrounding region. The arrows are oriented and scaled as per the directions and magnitudes of winds at different points.

Due to the combined effects of gravity and rotation, Jupiter is slightly flattened at its poles. The equation of a spheroid approximating for the shape of Jupiter can be stated as:

$$\frac{x^2 + y^2}{R_e^2} + \frac{z^2}{R_p^2} = 1,$$

where $R_e = 7.15 \times 10^7$ m is the equatorial radius of Jupiter, and $R_p = 6.69 \times 10^7$ m the polar radius. The radii of curvature of this spheroid in any direction can be calculated by the following equations ($\epsilon = \frac{R_e}{R_p}$):

$$r(\phi) = R_e \left(1 + \epsilon^{-2} \tan^2 \phi\right)^{-1/2}$$

$$R(\phi) = R_e \epsilon^{-2} \left(\frac{r(\phi)}{R_e \cos \phi}\right)^3$$

where $r$ and $R$ are the zonal (aka in the zone of a particular latitude) and meridional (aka longitudinal) radii of curvature, respectively, as a function of planetographic latitude $\phi$. The sidereal rotation period of Jupiter is $P = 3.57 \times 10^4$ s.

(a) **(4 points)** Calculate the zonal and meridional radii values ($\overline{r}$ and $\overline{R}$ respectively) at the location of the centre of the GRS.

(b) **(5 points)** Estimate the eccentricity of the GRS.

(c) **(6 points)** 'Vorticity' at any point is a measure of local spinning of the fluid as measured by an observer situated in the reference frame of the fluid. Mathematically, it is calculated as 'curl' (vector derivative product) of the velocity field. In this case, the average relative vorticity may be estimated by the equation:

$$\xi = \frac{V_w L_{GRS}}{A_{GRS}}$$

where $V_w$ is the maximum value of winds as per the velocity field, $L_{GRS}$ is the length of the circumference of the GRS and $A_{GRS}$ is the area of the GRS.
Estimate average relative vorticity of the GRS.
**Hint:** The circumference of an ellipse is well approximated by an average of circumferences of the corresponding auxiliary and minor circles.

(d) **(2 points)** Find the absolute vorticity $\xi_a = (\xi + f)$ by adding the Coriolis parameter

$$f = 2\Omega \sin \phi$$

where $\Omega$ is the angular velocity of the Jupiter (due to axial rotation) and is the appropriate latitude.

(e) **(1 point)** If the absolute vorticity has the same sign as the latitude, we call the storm a 'cyclonic storm'. If they have opposite signs, the system is 'anticyclonic'. Is the GRS cyclonic or anticyclonic?

(f) **(12 points)** Imagine that the GRS moves to another latitude $\phi_1$, where the absolute vorticity changes the sign (changes from anti-cyclonic to cyclonic or vice versa). Assuming minimum possible displacement of the GRS, at what value of $\phi_1$ do we expect this change?
In your analysis, assume that the GRS at the new location would occupy the same angular span in latitude, as well as have the same wind velocities and eccentricity as the original.

**Solution**

(a) From the figure, the centre of GRS lies at $\phi = -20.0°$. **1.0**

$$\begin{aligned}
\epsilon &= \frac{R_e}{R_p} = \frac{7.15}{6.69} \\
&= 1.069 && \textbf{1.0} \\
\overline{r} = r(-20°) &= R_e\left(1 + \epsilon^{-2}\tan^2\phi\right)^{-1/2} \\
&= 7.15 \times 10^7 \times \left(1 + (1.0688)^{-2}\tan^2(-20°)\right)^{-1/2} \\
\overline{r} &= 6.77 \times 10^7\,\mathrm{m} && \textbf{1.0} \\
\overline{R} = R(-20°) &= R_e\epsilon^{-2}\left(\frac{r(\phi)}{R_e\cos\phi}\right)^3 \\
&= 7.15 \times 10^7 \times (1.0688)^{-2}\left(\frac{r(-20°)}{7.15 \times 10^7 \times \cos(-20°)}\right)^3 \\
\overline{R} &= 6.40 \times 10^7\,\mathrm{m} && \textbf{1.0}
\end{aligned}$$

(b) We use the figure to estimate size major and minor axis of the GRS and hence estimate the eccentricity.

$$\begin{aligned}
a &= \overline{r}\theta_{long} \\
&= 6.77 \times 10^7 \times \frac{8.2° \times \pi}{180°} \\
a &= 9.69 \times 10^6\,\mathrm{m} && \textbf{2.0} \\
b &= \overline{R}\theta_{lat} \\
&= 6.40 \times 10^7 \times \frac{4.3° \times \pi}{180°} \\
b &= 4.80 \times 10^6\,\mathrm{m} && \textbf{2.0} \\
e_{GRS} &= \sqrt{1 - \left(\frac{b}{a}\right)^2} = \sqrt{1 - \left(\frac{4.8}{9.7}\right)^2} \\
e_{GRS} &\approx 0.87 && \textbf{1.0}
\end{aligned}$$

(c) The maximum value of wind velocity ($V_w$) is about $120\,\mathrm{m/s}$. **1.0**

$$\begin{aligned}
A_{GRS} &= \pi ab && \textbf{1.0} \\
L_{GRS} &\approx \frac{2\pi a + 2\pi b}{2} = \pi a(1 + \sqrt{1 - e^2}) && \textbf{2.0} \\
\therefore \xi &= \frac{V_w L_{GRS}}{A_{GRS}} = \frac{V_w \pi a(1 + \sqrt{1 - e^2})}{\pi ab} \\
&= \frac{120 \times (1 + \sqrt{1 - 0.87^2})}{4.8 \times 10^6} \\
\xi &\approx 3.7 \times 10^{-5}\,\mathrm{s}^{-1} && \textbf{2.0}
\end{aligned}$$

(d) The absolute vorticity will be given by

$$
\begin{aligned}
\xi_a &= \xi + 2\Omega \sin\phi \\
&= \xi + \frac{2 \times 2 \times \pi \sin -20^\circ}{P} = 3.7 \times 10^{-5} + \frac{4\pi \sin -20^\circ}{3.57 \times 10^4} \\
&= 3.7 \times 10^{-5} - 1.2 \times 10^{-4} \\
\xi_a &= -8.3 \times 10^{-5}
\end{aligned}
$$
**2.0**

(e) The GRS is a cyclonic system. **1.0**

(f) For this, we have to write $\xi_a$ as a function of $\phi$ and then find root of the function closest to the current latitude.

$$\xi_a = 0 = \xi + 2\Omega \sin\phi$$
**1.0**

$$
\begin{aligned}
0 &= \frac{V_w(1+\sqrt{1-e^2})}{b} + 2\Omega \sin\phi_1 \\
0 &= \frac{V_w(1+\sqrt{1-e^2})}{\overline{R}_1 \theta_{lat}} + 2\Omega \sin\phi_1 \\
0 &= \frac{V_w \epsilon^2 R_e^2 (1+\sqrt{1-e^2}) \cos^3\phi}{\theta_{lat} r^3(\phi_1)} + 2\Omega \sin\phi_1 \\
0 &= \frac{V_w \epsilon^2 R_e^2 (1+\sqrt{1-e^2}) \cos^3\phi_1}{\theta_{lat} (R_e\,(1+\epsilon^{-2}\tan^2\phi_1)^{-1/2})^3} + 2\Omega \sin\phi_1
\end{aligned}
$$
**3.0**

$$
\begin{aligned}
0 &= \left(\frac{V_w \epsilon^2 (1+\sqrt{1-e^2})}{\theta_{lat} R_e}\right)\left(\cos\phi_1 \sqrt{1+\epsilon^{-2}\tan^2\phi_1}\right)^3 + 2\Omega \sin\phi_1 \\
-3.520 \times 10^{-4} \sin\phi_1 &= 3.81 \times 10^{-5} \times \left(1 - (1-\epsilon^{-2})\sin^2\phi_1\right)^{3/2} \\
35.2^2 \sin^2\phi_1 &= 3.81^2 \times \left(1 - (1-\epsilon^{-2})\sin^2\phi_1\right)^3
\end{aligned}
$$

Let us define $\sin^2\phi_1 = x$

$$\therefore x = 0.01172\,(1 - 0.1245x)^3$$
**4.0**

Iterating over $x$, with starting value as $x_0 = \sin^2(-20^\circ) = 0.11698$

| $\mathbf{x_0}$ | $\mathbf{x_1}$ |
|---|---|
| 0.11698 | 0.01121 |
| 0.01121 | 0.01167 |
| 0.01167 | 0.01167 |

**2.0**

Thus,

$$
\begin{aligned}
\phi_1 &= -\sin^{-1}(\sqrt{0.01167}) \\
\phi_1 &= -6.2^\circ \approx -6^\circ
\end{aligned}
$$
**2.0**

An alternative solution is to try to find this using the plotting method, but one needs to make sure the result is obtained to the equivalent precision.